

Quantum theory on Lobatchevski spaces

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Abstract

In this paper we set up a general formalism to deal with quantum theories on a Lobatchevski space, i.e. a spatial manifold that is homogeneous, isotropic and has negative curvature. The heart of our approach is the construction of a suitable basis of plane waves which are eigenfunctions of the Laplace-Beltrami operator relative to the geometry of the curved space. These functions were previously introduced in the mathematical literature in the context of group theory; here we revisit and adapt the formalism in a way specific for quantum mechanics. Our developments render dealing with Lobatchevski spaces, which used to be quite difficult and source of controversies, easily tractable. Applications to the Milne and de Sitter universes are discussed as examples.

1 Introduction.

There are several good reasons to study quantum theories on a Lobatchevski space. The first reason is simply to extend our knowledge and skills in quantization. The negative curvature of this model together with its non compactness may, and indeed do, give rise to new and unforeseen phenomena which ordinary (canonical) quantization procedures on curved spacetimes [1] are not prepared to deal with.

Secondly, it has for long been advocated by Callan and Wilczek [2] that a negative curvature may act as a covariant regularizer of the infrared problem in a better way than putting a quantum system in a box or on a sphere. An intuitive way to understand this viewpoint is as follows: since in this geometry the volume of a sphere increases exponentially with its radius, the flux created by a central charge decreases accordingly and photons behave as if they have a "mass". Many quantum field models have indeed been studied on this background geometry, in particular the two-dimensional Liouville model [3] and a whole class of conformally invariant models.

From the viewpoint of astrophysics and cosmology, Friedmann-Robertson-Walker open models (more properly, models with negative spatial curvature), first introduced in the inflationary context in the early eighties [4], became rather popular [5, 6, 7, 8] in the mid-nineties, when the belief was that a negative curvature of the space could explain the mass content and the expansion rate of the universe. At that time it was realized that in some cases [9] there were troubles with canonical quantization in such a geometry, although with adaptation the latter can be used to recover the standard [10, 11, 12] Klein-Gordon quantum field in the special case of the open de Sitter manifold and some indications exist [13] for possible ways out in more general situations. One difficulty was the appearance of supercurvature modes for theories of sufficiently low mass. Those modes are not square integrable on the Lobatchevski manifold itself (even not in generalized sense as the exponentials of the flat case do) and therefore do not fit in the standard formalism of quantum mechanics. Even though the physical meaning of those supercurvature modes is yet unclear, they are not expected to provide a sizeable contribution to the

microwave background fluctuations [14]. They might however be relevant in other domains of physics ... and even in cosmology that from time to time (and even often) gives rise to surprises.

The interest in open inflationary models subsequently dropped with the measurements of the fluctuations of the Cosmic Microwave Background [15] indicating that the universe is most likely flat. However, with the spectacular and unprecedented precision of the forthcoming measurements scheduled by the PLANCK satellite [16], any small deviation from flatness will have to be mastered securely. This calls for a revision and a complete solution of the problem of describing quantum fluctuations in a negatively curved space (while the positively curved case is completely under control).

From the mathematical viewpoint, part of the material we are going to present belongs to the chapter of harmonic analysis on symmetric spaces (see e.g. [17]; [18] gives an up-to-date account of the topics). Of course there exists a vast literature already on the more specific subject of Lobatchevski spaces. These mathematical results however are often formulated in a rather abstract way. Also, the square-integrability hypothesis that is a pillar of harmonic analysis may be violated in physically interesting situations and should only be considered as a starting point. On the other hand, most if not all approaches in the physical literature involve series expansions in terms of special functions which hide to a large extent the underlying symmetries and ultimately the physics.

We have therefore decided to reconsider the subject from the beginning and we have found some new ways to handle efficiently, and rather simply, the technical difficulties that arise from the lack of a commutative group of space translations. Our approach is especially aimed for physics; it remains practical and accessible and has also the advantage to avoid the use of theorems concerning expansions in bases of special functions (those expansions actually result from our approach). Another valuable point is that we work solely within the physical space-time and do not rely on any extension or completion of the latter to regions that are not covered by the open chart, as is the case in previous studies. Our scheme is sufficiently flexible to allow the study of the general dimension in a single step.

We focus on the study of a basis for the the standard Hilbert space of square-integrable functions on a $(d - 1)$ -dimensional Lobatchevski space (so that the spacetime has d dimensions). This Hilbert space is not expected to be sufficient [9, 13] to describe all the interesting physical quantum theories in case of negative spatial curvature. Modes that are not square-integrable, however, warrant a separate specific study, and will be considered elsewhere [19] along the same line of thought.

In Section 2 we give an introductory presentation of the geometry of the Lobatchevski space and of the "absolute" [20] of that space which can be identified with the space of momentum directions. We give also a quite detailed description of the coordinate systems we are going to use.

In Section 3 we display the unnormalized eigenmodes of the Laplace-Beltrami operator following the approach described in [20]. These solutions are "plane wavefunctions" in the sense they have constant values on hyperplanes. Strangely enough, these solutions have been only rarely used to study quantum theories in a Lobatchevski space in the physical literature, but they constitute the most natural possibility, and actually a cornerstone, to phrase Lobatchevskian quantum mechanics in strict analogy with Euclidean quantum mechanics. An important technical point is provided in Section 4 where we construct some useful integral representations of the above eigenmodes; these representations are linked to the parabolic coordinate system introduced in Section 2 and are inspired by our earlier work [21]. These representations allow to trivialize $(d - 2)$ integrations when a square-integrable wavefunction is projected on a mode. The relation of our modes with the basis constructed in spherical coordinates (see e.g. [5] and references therein) is also discussed.

In Sections 5, 6 and 7 we set up the basic ingredients for quantum mechanics by building an orthonormal basis of modes; in particular we show how to deal with the integral representations by computing the L^2 scalar product of the modes. This naturally leads to the introduction of a Fourier-type transform of an arbitrary L^2 function and to an inversion formula obtained here by means of the Kontorovich-Lebedev transform.

We conclude the paper by discussing two physical applications to models of QFT that are playing a central role in contemporary cosmology.

In Section 8 we revisit Milne QFT. There has been a regain of interest in this model in recent times coming from the very different perspectives of string theory [22, 3] and of observational cosmology [23]. Here we give a clean treatment of Milne's QFT based on an expansion of the Minkowskian exponential plane waves onto the basis of Lobatchevski modes that we have been constructing.

In the following Section 9, we study QFT on the open de Sitter universe at any dimension. This has proven to be technically very difficult and has been source of controversies, even for the quantification in L^2 space, which is the aim of the present paper [9, 13]. A special attention is given to the two dimensional case, which shares many features of the general case, but is extremely simple.

The anti-de Sitter case is of obvious interest: the Lobatchevski manifold is identical to the Euclidean anti-de Sitter universe. The physics however is however very different and therefore we have not included the AdS case in the present paper.

2 Coordinates.

In this section we describe the relevant geometrical setup for our construction, following the ideas and the layout of Gel'fand, Graev and Vilenkin [20].

Let \mathbb{M}^d be a d -dimensional Minkowski spacetime; an event x has inertial coordinates x^0, \dots, x^{d-1} and the scalar product of two such events is given by

$$x \cdot x' = x^0 x'^0 - x^1 x'^1 - \dots - x^{d-1} x'^{d-1} = \eta_{\mu\nu} x^\mu x'^\nu. \quad (1)$$

In \mathbb{M}^d we consider the manifold (Fig. 1)

$$\mathbb{H}^{d-1} = \{x \in \mathbb{M}^d : x^2 = x \cdot x = 1, x^0 > 0\}. \quad (2)$$

The manifold \mathbb{H}^{d-1} models a $(d-1)$ -dimensional Lobatchevski hyperbolic space¹. It is a homogeneous space under the action of the restricted Lorentz group of the ambient spacetime $SO_0(1, d-1)$. The Riemannian metric dl_{d-1}^2 is obtained by restriction of the ambient Lorentzian metric to \mathbb{H}^{d-1} :

$$dl_{d-1}^2 = - \left\{ (dx^0)^2 - (dx^1)^2 - \dots - (dx^{d-1})^2 \right\} \Big|_{\mathbb{H}^{d-1}}. \quad (3)$$

2.1 Parabolic coordinates.

Many interesting coordinate systems on \mathbb{H}^{d-1} arise from particular decompositions of the symmetry group $SO_0(1, d-1)$. In this paper we will mainly use the following one:

$$x(r, \mathbf{x}) = \begin{cases} x^0 &= \frac{1+\mathbf{x}^2+r^2}{2r} \\ x^i &= \frac{x^i}{r} \\ x^{d-1} &= \frac{1-\mathbf{x}^2-r^2}{2r} \end{cases}, \quad 0 < r < \infty, \quad \mathbf{x} \in \mathbb{R}^{d-2}. \quad (4)$$

In these coordinates the metric and the invariant volume form have the following explicit expressions:

$$dl_{d-1}^2 = \frac{dr^2 + d\mathbf{x}^2}{r^2}, \quad d\mu(x) = r^{-(d-2)} \frac{dr}{r} d\mathbf{x}, \quad (5)$$

¹We use here the convention mostly adopted by topologists in using the notation \mathbb{H}^{d-1} . The index $d-1$ refers to the actual dimension of the manifold in the same way as $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d, x^1^2 + \dots x^{d^2} = 1\}$ denotes a $(d-1)$ -dimensional sphere embedded in a space of dimension d . Note however that the conventional notation for the hypersurface of the sphere \mathbb{S}^{d-1} is $\omega_d = 2\pi^{d/2}/\Gamma(d/2)$.

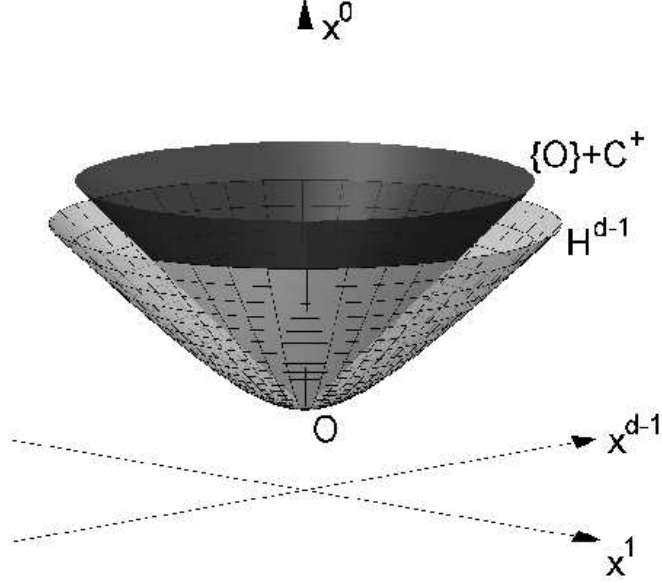


Figure 1: A space of constant negative curvature embedded in an ambient Minkowski spacetime with one dimension more. The manifold \mathbb{H}^{d-1} does not intersect the lightcone issued at any of its points (the origin O in the figure); this shows pictorially that the surface is a spacelike manifold i.e. a Riemannian model of space.

while the scalar product of two event is written

$$x(r, \mathbf{x}) \cdot x'(r', \mathbf{x}') = \frac{(\mathbf{x} - \mathbf{x}')^2 + r^2 + r'^2}{2rr'}. \quad (6)$$

Equations (5) and (6) show that the measure and the scalar product are invariant w.r.t. translations in the \mathbf{x} coordinates and this explain why this system is so important and useful. Dirac's delta distribution on the hyperboloid \mathbb{H}^{d-1} is understood w.r.t. the invariant measure

$$\int_{\mathbb{H}^{d-1}} d\mu(x') \delta(x, x') f(x') = f(x); \quad (7)$$

specifically, in the previous coordinate system it is written as follows:

$$\delta(x, x') = r^{d-1} \delta(r - r') \delta(\mathbf{x} - \mathbf{x}'). \quad (8)$$

The future light-cone of the ambient spacetime (Fig. 2) plays a crucial role in our construction:

$$C^+ = \{\xi^2 = \xi \cdot \xi = 0, \quad \xi^0 > 0\}. \quad (9)$$

A useful parametrization for vectors of C^+ corresponding to (4) is the following:

$$\xi(\lambda, \eta) = \begin{cases} \xi^0 &= \frac{\lambda}{2}(1 + \eta^2) \\ \xi^i &= \lambda \eta^i \\ \xi^{d-1} &= \frac{\lambda}{2}(1 - \eta^2) \end{cases} \quad 0 < \lambda < \infty, \quad \eta \in \mathbb{R}^{d-2}. \quad (10)$$

From both the mathematical and the physical viewpoints what matters is [20] the "absolute" of the Lobatchevski space \mathbb{H}^{d-1} : this is the set of linear generators of the future light-cone, i.e. the light-cone

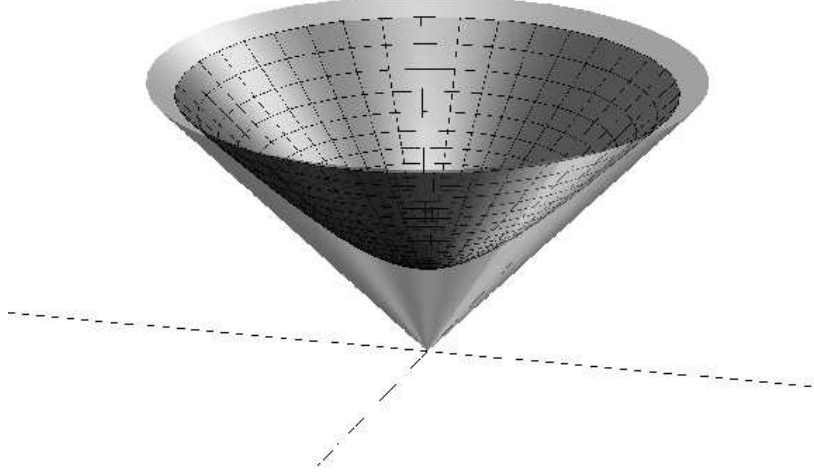


Figure 2: *A sight of the asymptotic future lightcone of the ambient spacetime; the vectors ξ belonging to C^+ play the role of momentum directions.*

modulo dilatations. The previous parametrization gives in particular a parabolic parametrization of the absolute that is visualized (Fig. 3) as the parabolic section $\lambda = 1$ of C^+ :

$$\xi(\eta) \equiv \xi(1, \eta) = \begin{cases} \xi^0 &= \frac{1}{2}(1 + \eta^2) \\ \xi^i &= \eta^i \\ \xi^{d-1} &= \frac{1}{2}(1 - \eta^2) \end{cases} \quad \eta \in \mathbb{R}^{d-2}. \quad (11)$$

Correspondingly, the measure on the absolute is normalized as follows:

$$d\mu(\xi) = d\eta. \quad (12)$$

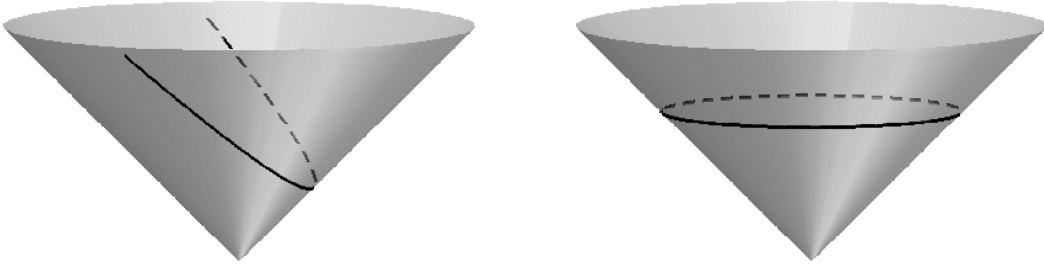


Figure 3: *A view of the parabolic and spherical bases of the absolute.*

Dirac's delta on the cone will actually mean Dirac's delta on the absolute:

$$\delta(\xi, \xi') = \delta(\eta - \eta'). \quad (13)$$

By using the coordinate systems (4) and (11) the scalar products $x \cdot \xi$ and $\xi \cdot \xi'$ are written as fol-

lows:

$$x(r, \mathbf{x}) \cdot \xi(\eta) = \frac{(\mathbf{x} - \eta)^2}{2r} + \frac{r}{2} = \frac{\xi(\mathbf{x}) \cdot \xi(\eta)}{r} + \frac{r}{2} \quad (14)$$

$$\xi(\eta) \cdot \xi'(\eta') = \frac{(\eta - \eta')^2}{2} \quad (15)$$

where we have introduced the lightlike vector $\xi(\mathbf{x})$ corresponding to $x(r, \mathbf{x})$. Note that this correspondence is tight to the choice of coordinates (4). Starting from $\xi(\mathbf{x})$ we may recover the point x as follows (Fig. 4):

$$x(r, \mathbf{x}) = \frac{\xi(\mathbf{x})}{r} + r\hat{\xi} \quad (16)$$

where $\hat{\xi} = \xi(0) = (\frac{1}{2}, 0, \dots, -\frac{1}{2})$ generates the only lightlike direction that the parametrization (11) does not cover; this direction is actually attained in the limit $r \rightarrow \infty$ as shown by Eq. (16); one has that $\hat{\xi} \cdot x(r, \mathbf{x}) = 1/2r$.

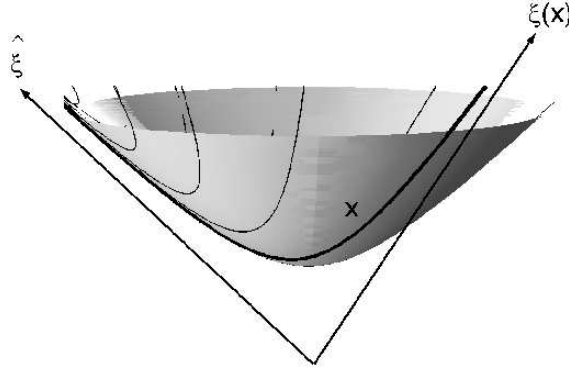


Figure 4: The choice of the vector $\hat{\xi}$, given the point x , uniquely determines the vector $\xi(\mathbf{x})$. The latter is the vector obtained by intersecting the cone with the plane containing the vector $\hat{\xi}$ and the point x (this plane actually contains the whole curve $x(r, \mathbf{x})$, $r > 0$). "Infinity" on the curve $x(r, \mathbf{x})$ is at $r = 0$ in the direction $\xi(\mathbf{x})$ - which depend on the chosen point \mathbf{x} - and at $r = \infty$ in the direction $\hat{\xi}$.

2.2 Spherical coordinates.

There is the other natural and widely used coordinate system on the manifold \mathbb{H}^{d-1} where the rotation symmetry $SO(d-1)$ is apparent:

$$x(\psi, \phi) = \begin{cases} x^0 = \cosh \psi \\ x^1 = \sinh \psi \cos \phi^1 \\ \vdots \\ x^{d-2} = \sinh \psi \sin \phi^1 \dots \sin \phi^{d-3} \cos \phi^{d-2} \\ x^{d-1} = \sinh \psi \sin \phi^1 \dots \sin \phi^{d-3} \sin \phi^{d-2} \end{cases} \quad (17)$$

$\psi \in \mathbb{R}, \quad \phi = (\phi^1, \dots, \phi^{d-2}), \quad 0 < \phi^i < \pi, \quad 0 < \phi^{d-2} < 2\pi.$

Correspondingly, there is the important (compact) spherical basis $\xi^0 = 1$ for the absolute (see Fig. (3)):

$$\xi(\theta) = \begin{cases} \xi^0 = 1 \\ \vec{\xi} = \mathbf{n}_\theta \end{cases} = \begin{cases} \xi^0 = 1 \\ \xi^1 = \cos \theta^1 \\ \vdots \\ \xi^{d-2} = \sin \theta^1 \dots \sin \theta^{d-3} \cos \theta^{d-2} \\ \xi^{d-1} = \sin \theta^1 \dots \sin \theta^{d-3} \sin \theta^{d-2} \end{cases} \quad (18)$$

with $\theta = (\theta^1, \dots, \theta^{d-2})$ and where \mathbf{n}_θ is a unit vector ($\mathbf{n}_\theta \cdot \mathbf{n}_\theta = 1$) pointing in the direction identified by the angles θ . In these coordinates the scalar products are written as follows

$$x(\psi, \phi) \cdot x(\psi', \phi') = \cosh \psi \cosh \psi' - \sinh \psi \sinh \psi' \mathbf{n}_\phi \cdot \mathbf{n}_{\phi'} \quad (19)$$

$$x(\psi, \phi) \cdot \xi(\theta) = \cosh \psi - \sinh \psi \mathbf{n}_\phi \cdot \mathbf{n}_\theta \quad (20)$$

$$\xi(\theta) \cdot \xi(\theta') = 1 - \mathbf{n}_\theta \cdot \mathbf{n}_{\theta'} \quad (21)$$

The measure on the absolute is the rotation invariant measure normalised as follows:

$$d\mu(\theta) = \prod_{i=1}^{d-2} (\sin \theta^i)^{d-2-i} d\theta^i$$

3 Modes of the Laplacian. Plane waves.

Let us consider the vector space $\mathcal{S}(\mathbb{H}^{d-1})$ of rapidly decreasing complex functions defined on the manifold \mathbb{H}^{d-1} and let us introduce the natural scalar product w.r.t. the invariant measure $d\mu(x)$:

$$\langle f, g \rangle = \int_{\mathbb{H}^{d-1}} f^*(x) g(x) d\mu(x) . \quad (22)$$

The space $\mathcal{S}(\mathbb{H}^{d-1})$ can be completed to construct the Hilbert space $\mathcal{H} = L^2(\mathbb{H}^{d-1}, d\mu)$. By analogy with the flat case, \mathcal{H} is the natural Hilbert space one would consider to study quantum mechanics on the homogeneous and isotropic hyperbolic space \mathbb{H}^{d-1} . To pursue this analogy, the first object to be examined is the free Hamiltonian operator

$$H = -\Delta , \quad (23)$$

where Δ denotes the Laplace-Beltrami operator associated with the geometry (3). The operator $-\Delta$ is self-adjoint on a suitable domain of the Hilbert space \mathcal{H} and its spectrum is the set $[\frac{1}{4}(d-2)^2, \infty)$. The eigenfunctions of the operator $-\Delta$ can be labeled by a forward lightlike vector $\xi \in C^+$ and a real number q as follows [20]:

$$\psi_{iq}(x, \xi) = \text{const}(x \cdot \xi)^{-\frac{d-2}{2}+iq} , \quad (24)$$

and one easily verifies that ²

$$-\Delta\psi_{iq}(x, \xi) = \left[\left(\frac{d-2}{2} \right)^2 + q^2 \right] \psi_{iq}(x, \xi) = k^2 \psi_{iq}(x, \xi). \quad (28)$$

For fixed q and any $\lambda > 0$, the vectors ξ and $\lambda\xi$ identify the same eigenfunction because of the homogeneity properties of the expression (24). Therefore the modes (24) corresponding to real values of the parameter q and to vectors ξ on the absolute (i.e. on a basis of the asymptotic lightcone) can be used to construct a basis of the Hilbert space \mathcal{H} and are the strict analogue of the purely imaginary exponentials $e^{ip \cdot x}$ of the flat case. Indeed, like the exponentials, also the wavefunctions (24) take constant values on planes, here the hyperplanes $x \cdot \xi = \text{const.}$ In this sense the modes (24) may be called “plane waves”.

As for the physical interpretation of the labels, k may be thought of as the intensity and ξ as the “direction” of a “momentum” vector identifying a mode. The analogy in flat space would be parameterizing the plane wave by the modulus of the momentum $p = (p \cdot p)^{\frac{1}{2}}$ and by its direction n : $e^{ip \cdot n \cdot x}$.

This analogy goes one step further: we could have considered another set of plane waves characterized by the opposite of the modulus of the momentum $-p$ and by a direction n' : $e^{-ip \cdot n' \cdot x}$. A trivial remark is that these modes already belong to the set $\{e^{ip \cdot n \cdot x}\}$ since n can point in any direction.

We will see in the following section, that also the mode $(x \cdot \xi)^{-\frac{d-2}{2}-iq}$ can be expressed as a superposition on the absolute (i.e. as an integral over ξ' on a basis of the cone C^+) of the modes $(x \cdot \xi')^{-\frac{d-2}{2}+iq}$. This means that to construct a basis of the Hilbert space \mathcal{H} we may restrict our attention to the plane waves $\psi_{iq}(x, \xi)$, $q \geq 0$, ξ on the absolute.

We end this section by remarking that there are purely imaginary values of q such that $k^2 \geq 0$, namely those purely imaginary q such that $|q| \leq \frac{d-1}{2}$. In particular the mode corresponding to $k = 0$ is constant in space. These waves are not conventional in many respects; they are real functions, do not oscillate and (superposition of them) do not belong to the natural Hilbert space \mathcal{H} . This means that a standard quantum mechanical interpretation is not immediately available for them. Such cases have been considered in the past [9, 13] but their status is not completely understood. We will discuss their possible role elsewhere [19].

4 Representations of the modes.

There is no way to write the modes of the Laplace-Beltrami operator that be more symmetric than the expression (24): in that definition the modes appear as a complex power of a quantity invariant under the action of the symmetry group, exactly as it happens for the exponentials $\exp(ip \cdot x)$ in the flat case. The exponentials have however the important property to be characters of the translation

²This can be shown by using a specific coordinate system, for instance (4), or either by using the inertial coordinates of the ambient spacetime, by introducing the projection operator h and the tangential derivative D as follows:

$$h^{\mu\nu} = \eta^{\mu\nu} - x^\mu x^\nu, \quad D^\mu = h^{\mu\nu} \partial_\nu = \partial^\mu - x^\mu x \cdot \partial. \quad (25)$$

For any function that is smooth in a neighborhood of the manifold \mathbb{H}^{d-1} one has that

$$D^\mu D_\mu f = \square f - (d-2) x \cdot \partial f - x \cdot \partial (x \cdot \partial f), \quad (26)$$

where \square is the wave operator in the d -dimensional ambient spacetime. From this relation one can easily see that

$$D^\mu D_\mu (x \cdot \xi)^\alpha = \alpha(\alpha-1)\xi^2 (x \cdot \xi)^\alpha - \alpha(\alpha+d-2) (x \cdot \xi)^\alpha = -\alpha(\alpha+d-2) (x \cdot \xi)^\alpha \quad (27)$$

Since $-\Delta f = D^\mu D_\mu \hat{f} \Big|_{\mathbb{H}^{d-1}}$, where \hat{f} is any smooth extension of the function f in a neighborhood of the manifold \mathbb{H}^{d-1} Eq. (28) follows.

group: this property is expressed by the relation $\exp(ip \cdot x) \exp(ip \cdot y) = \exp(ip \cdot (x + y))$. The lack of translation invariance of \mathbb{H}^{d-1} is a major technical difficulty and, for our modes, there is nothing immediately replacing this property of the exponentials. Therefore, from the viewpoint of practical calculations, it is useful to represent the modes (24) in terms of some integral transform which be reminiscent of translational invariance. Many representations are possible, in relation with different choices of coordinates on \mathbb{H}^{d-1} and on the cone; we list only the two that are more relevant for our purposes.

4.1 Euler Representation.

The following integral representations is an adaptation of the Euler integral of the second kind expressing the Gamma function; an interesting geometrical interpretation can be based on the embedding of \mathbb{H}^{d-1} in the ambient Minkowski spacetime \mathbb{M}^d :

$$(x \cdot \xi)^{-\frac{d-2}{2}+iq} = \frac{1}{\Gamma\left(\frac{d-2}{2} - iq\right)} \int_0^\infty \frac{dR}{R} R^{\frac{d-2}{2}-iq} e^{-Rx \cdot \xi}. \quad (29)$$

This representation is valid if $\text{Im } q > \text{Re } \frac{d-2}{2}$ and $\text{Re}(x \cdot \xi) > 0$. More generally one can perform the integration on the complex plane as follows:

$$(x \cdot \xi)^{-\frac{d-2}{2}+iq} = \frac{i^{-(\frac{d-2}{2}-iq)}}{\Gamma\left(\frac{d-2}{2} - iq\right)} \int \frac{dR}{R} R^{\frac{d-2}{2}-iq} e^{iRx \cdot \xi} \quad (30)$$

where the integration contour in the complex R plane is along any half-line issued from the origin in the upper half-plane, i.e. $0 < \text{Arg}(R) < \pi$.

4.2 Fourier Representations.

Another useful integral representations can be obtained by inserting at the RHS of Eq. (29) the representation (14) of the scalar product $x \cdot \xi$ and Fourier transforming the Gaussian factor appearing there:

$$\begin{aligned} (x \cdot \xi)^{-\frac{d-2}{2}+iq} &= \frac{1}{\Gamma\left(\frac{d-2}{2} - iq\right)} \left(\frac{r}{2\pi}\right)^{\frac{d-2}{2}} \int \frac{d\kappa}{\kappa^{iq}} e^{-i\kappa \cdot (x-\eta)} \int_0^\infty \frac{dR}{R} \frac{e^{-\frac{1}{2}r\kappa(\frac{1}{R}+R)}}{R^{iq}} \\ &= \frac{2}{\Gamma\left(\frac{d-2}{2} - iq\right)} \left(\frac{r}{2\pi}\right)^{\frac{d-2}{2}} \int d\kappa \kappa^{-iq} e^{-i\kappa \cdot (x-\eta)} K_{iq}(\kappa r) \end{aligned} \quad (31)$$

where $\kappa = \sqrt{\kappa \cdot \kappa}$. In the second step we have used a well-known integral representation of the Bessel-Macdonald function $K_{iq}(z)$ that we recall here for the reader's convenience:

$$K_{iq}(z) = \frac{1}{2} \int_0^\infty \frac{dR}{R} R^{-iq} e^{-\frac{1}{2}z(R+\frac{1}{R})}. \quad (32)$$

The function $K_{iq}(z)$ decreases exponentially at large z ; near the origin one has that $K_{iq}(z) \sim \frac{\Gamma(iq)}{2} \left(\frac{z}{2}\right)^{-iq} + \frac{\Gamma(-iq)}{2} \left(\frac{z}{2}\right)^{iq}$. Therefore the integral (31) converges at $\kappa = 0$ and provides a representation of $(x \cdot \xi)^{-\frac{d-2}{2}+iq}$ for $|\text{Im } q| < \frac{d-2}{2}$.

4.3 Expansion in spherical harmonics.

There exists an abundant literature on Lobatchevski spaces that is based on the use of generalized spherical harmonics in connection with spherical coordinate system. To render possible a comparison of our results and methods with that approach let us work out the change of basis.

By adopting the spherical coordinates of Section 2.2 we can write (see Eq. 20)

$$e^{iR x \cdot \xi} = e^{iR \cosh \psi} e^{-iR \sinh \psi \mathbf{n}_\phi \cdot \mathbf{n}_\theta}. \quad (33)$$

The spherical leaves of our $(d-1)$ -dimensional Lobatchevski space have $(d-2)$ dimensions and we may expand the second factor at the RHS in terms of generalized spherical harmonics $Y_{l,M}(\mathbf{n}_\phi)$ depending on $(d-2)$ angles, where M is a multi-index encoding $(d-3)$ “magnetic” indices in addition to l ; in the standard two-dimensional case (that corresponds to $d=4$) $Y_{l,M}(\mathbf{n}_\phi)$ are the standard spherical harmonics $Y_{l,m}(\phi^1, \phi^2)$. The starting point is as usual the well-known expansion of the exponential in terms of Gegenbauer polynomials C_l^ν and Bessel functions (see e.g. [24] Eq. (7.10;5)):

$$e^{i\gamma z} = \left(\frac{2}{z}\right)^\nu \Gamma(\nu) \sum_{l=0}^{\infty} i^l (\nu+l) C_l^\nu(\gamma) J_{\nu+l}(z). \quad (34)$$

The second step is to set ν to $\frac{d-3}{2}$ and take advantage of the known (see e.g. [25] Eq. (B.12)) expansion of $C_l^\nu(\mathbf{n}_\phi \cdot \mathbf{n}_\theta)$ as a sum of the generalized spherical harmonics:

$$e^{-iR \sinh \psi \mathbf{n}_\phi \cdot \mathbf{n}_\theta} = (2\pi)^{\frac{d-1}{2}} \sum_{l=0}^{\infty} i^{-l} (R \sinh \psi)^{-\frac{d-3}{2}} J_{l+\frac{d-3}{2}}(R \sinh \psi) \sum_M Y_{l,M}(\mathbf{n}_\theta) Y_{l,M}^*(\mathbf{n}_\phi). \quad (35)$$

By inserting this expression in the Euler representation (30) we get that

$$\begin{aligned} (x \cdot \xi)^{-\frac{d-2}{2}+iq} &= 2\pi \left(\frac{2\pi}{\sinh \psi}\right)^{\frac{d-3}{2}} \sum_{l=0}^{\infty} \frac{i^{-(l+\frac{d-2}{2}-iq)}}{\Gamma(\frac{d-2}{2}-iq)} \sum_M Y_{l,M}(\mathbf{n}_\theta) Y_{l,M}^*(\mathbf{n}_\phi) \\ &\quad \int_0^\infty \frac{dR}{R} R^{\frac{1}{2}-iq} e^{iR \cosh \psi} J_{l+\frac{d-3}{2}}(R \sinh \psi). \end{aligned} \quad (36)$$

The integral at the RHS is the Mellin transform of a product that can be evaluated by the Mellin-Barnes integral. This is a way to directly check [24] Eq. (7.8;9):

$$\int_0^\infty \frac{dR}{R} R^{\frac{1}{2}-iq} e^{iR \cosh \psi} J_{l+\frac{d-3}{2}}(R \sinh \psi) = i^{l+\frac{d-2}{2}-iq} \Gamma\left(l + \frac{d-2}{2} - iq\right) P_{-\frac{1}{2}+iq}^{-l-\frac{d-3}{2}}(\cosh \psi) \quad (37)$$

that holds for $\text{Re}(l + \frac{d-2}{2} - iq) > 0$, that is for all $l \geq 0$ provided³ $\text{Im } q > -\frac{d-2}{2}$. Therefore

$$\begin{aligned} (x(\psi, \phi) \cdot \xi(\theta))^{-\frac{d-2}{2}+iq} &= \\ &= 2\pi \left(\frac{2\pi}{\sinh \psi}\right)^{\frac{d-3}{2}} \sum_{l=0}^{\infty} \frac{\Gamma(l + \frac{d-2}{2} - iq)}{\Gamma(\frac{d-2}{2} - iq)} P_{-\frac{1}{2}+iq}^{-l-\frac{d-3}{2}}(\cosh \psi) \sum_M Y_{l,M}^*(\mathbf{n}_\theta) Y_{l,M}(\mathbf{n}_\phi) = \\ &= a(q) \sum_{l=0}^{\infty} \sum_M Y_{l,M}^*(\mathbf{n}_\theta) Z_{iq,l,M}(\psi, \mathbf{n}_\phi) \end{aligned} \quad (38)$$

where

$$a(q) = (2\pi)^{\frac{d-1}{2}} \frac{\Gamma(-iq)}{\Gamma(\frac{d-2}{2} - iq)}. \quad (39)$$

and

$$Z_{iq,l,M}(\psi, \mathbf{n}_\phi) = \frac{\Gamma(l + \frac{d-2}{2} - iq)}{\Gamma(-iq)} (\sinh \psi)^{-\frac{d-3}{2}} P_{-\frac{1}{2}+iq}^{-l-\frac{d-3}{2}}(\cosh \psi) Y_{l,M}(\mathbf{n}_\phi). \quad (40)$$

For $q \geq 0$, the latter are the orthonormal eigenmodes (see e.g. [5]; note a slight modification in our definition of Z) associated with the spherical coordinates.

³If we require $(x \cdot \xi)^{-\frac{d-2}{2}-iq}$ to bear simultaneously the same expansion, the condition of validity becomes $|\text{Im } q| < \frac{d-2}{2}$.

4.4 Asymptotics.

Using the coordinates (4) the "boundary" at infinity of the manifold \mathbb{H}^{d-1} is attained for either $r \rightarrow 0$ or $r \rightarrow \infty$ and the behaviour of the modes in these limits is of importance. Actually, only the behaviour at small r matters while the limit $r \rightarrow \infty$ is rather an artifact of the coordinate system (see Fig. 4). Eq. (31) together with the behaviour of $K_{iq}(\kappa r)$ at small r give the following asymptotics:

$$(x(r, x) \cdot \xi)^{-\frac{d-2}{2}+iq} \sim r^{\frac{d-2}{2}-iq} (\xi(x) \cdot \xi)^{-\frac{d-2}{2}+iq} + A(q) r^{\frac{d-2}{2}+iq} \delta(\xi(x) - \xi). \quad (41)$$

where

$$A(q) = \frac{(2\pi)^{\frac{d-2}{2}} 2^{-iq} \Gamma(-iq)}{\Gamma(\frac{d-2}{2} - iq)}. \quad (42)$$

By sending the point x that appears in Eq. (24) to the "boundary" at infinity of the manifold \mathbb{H}^{d-1} one gets naturally the two-point kernel $(\xi \cdot \xi')^{-\frac{d-2}{2}+iq}$ on the asymptotic cone. As before, this kernel admits useful Euler and Fourier representations:

$$(\xi \cdot \xi')^{-\frac{d-2}{2}+iq} = \frac{1}{\Gamma(\frac{d-2}{2} - iq)} \int_0^\infty \frac{dR}{R} e^{-R\xi \cdot \xi'} R^{\frac{d-2}{2}-iq} \quad (43)$$

$$= \frac{2^{iq} \Gamma(iq)}{(2\pi)^{\frac{d-2}{2}} \Gamma(\frac{d-2}{2} - iq)} \int d\kappa \kappa^{-2iq} e^{-i\kappa \cdot (\eta - \eta')}. \quad (44)$$

Let us discuss an immediate application of this definition by establishing the relation between modes with negative and positive values of q . Suppose that $(x \cdot \xi)^{-\frac{d-2}{2}-iq}$ be superposition of the modes $(x \cdot \xi')^{-\frac{d-2}{2}+iq}$, with $q > 0$.

Homogeneity implies that $(x \cdot \xi)^{-\frac{d-2}{2}-iq}$ and $\int d\mu(\xi') (\xi \cdot \xi')^{-\frac{d-2}{2}-iq} (x \cdot \xi')^{-\frac{d-2}{2}+iq}$ are proportional. This can be explicitly shown by using the integral representations (31) and (44) in order to perform the latter integration:

$$(x \cdot \xi)^{-\frac{d-2}{2}-iq} = \frac{1}{A(q)} \int d\mu(\xi') (\xi \cdot \xi')^{-\frac{d-2}{2}-iq} (x \cdot \xi')^{-\frac{d-2}{2}+iq} \quad (45)$$

5 Orthogonality of the modes on \mathbb{H}^{d-1} .

We are now ready to construct a basis for the Hilbert space $L^2(\mathbb{H}^{d-1}, d\mu)$ that can be used to study "ordinary" quantum theories on the Lobachevski space \mathbb{H}^{d-1} . The word ordinary refers to theories where the common wisdom and the standard tools of quantum mechanics, including the probabilistic interpretation, apply. As we have already said, there are also non-standard theories, corresponding to the allowed imaginary values of q . These theories will be examined elsewhere.

5.1 Scalar product of waves

The modes (24) correspond to eigenvalues of the continuous spectrum of the Laplacian. Of course modes corresponding to distinct values of q^2 are orthogonal, because of the self-adjointness of the Laplacian. To find the correct normalization we study the distributional kernel constructed by taking the scalar product of two modes (24) for q and q' are real. It follows that (see Section 5.3)

$$F_{q,q'}(\xi, \xi') = \int d\mu(x) (x \cdot \xi)^{-\frac{d-2}{2}-iq} (x \cdot \xi')^{-\frac{d-2}{2}+iq'} =$$

$$= \frac{1}{n(q)} \left[\delta(q - q') \delta(\eta - \eta') + \frac{1}{A(q)} (\xi \cdot \xi')^{-\frac{d-2}{2}-iq} \delta(q + q') \right] \quad (46)$$

while the normalization reads

$$\frac{1}{n(q)} = a(q)a(-q) = 2\pi A(q)A(-q) = \frac{(2\pi)^{d-1} \Gamma(iq) \Gamma(-iq)}{\Gamma\left(\frac{d-2}{2} + iq\right) \Gamma\left(\frac{d-2}{2} - iq\right)}. \quad (47)$$

We may recall that $a(q)$ is given in (39) and $A(q)$ in (42). This result is indeed consistent with (45).

5.2 Orthonormalized modes

It is appropriate to introduce normalized modes and the conjugate ones as follows. For $q \geq 0$, let us define

$$\psi_{iq}(x, \xi) = \frac{(x \cdot \xi)^{-\frac{d-2}{2}+iq}}{a(q)}, \quad \psi_{iq}^*(x, \xi) = \frac{(x \cdot \xi)^{-\frac{d-2}{2}-iq}}{a(-q)} \quad (48)$$

so that

$$\int d\mu(x) \psi_{iq}^*(x, \xi) \psi_{iq'}(x, \xi') = \delta(q - q') \delta(\xi, \xi'). \quad (49)$$

The set $\{\psi_{iq}(x, \xi), q \geq 0, \xi \text{ on the absolute}\}$ is then an orthonormal⁴ family of modes for the Hilbert space $L^2(\mathbb{H}^{d-1}, d\mu)$. The relation of these waves with the more commonly used waves in spherical coordinates is given by Eq. (38). The following suggestive expansion in term of generalized spherical harmonics is worth to be mentioned:

$$\psi_{iq}(x(\psi, \phi), \xi(\theta)) = \sum_{l=0}^{\infty} \sum_M Y_{l,M}^*(\mathbf{n}_\theta) Z_{iq,l,M}(\psi, \mathbf{n}_\phi) \quad (50)$$

and conversely

$$Z_{iq,l,M}(\psi, \mathbf{n}_\phi) = \int Y_{l,M}(\mathbf{n}_\theta) \psi_{iq}(x(\psi, \phi), \xi(\theta)) d\mu(\theta) \quad (51)$$

Our normalized plane waves are therefore superpositions of the spherical waves $Z_{iq,l,M}(\psi, \mathbf{n}_\phi)$ with weights which are themselves normalized generalized spherical harmonics evaluated at the direction of the vector $\xi(\theta)$ parametrizing the plane wave itself.

The advantage of the waves (48) is their independence on the choice of particular coordinate systems and, above all, their maximal symmetry. They really encode the symmetry of the Lobatchevski space. Their representations in terms of exponentials given in Sect. (4) also renders feasible calculations that are otherwise intractable (see below and [19]).

5.3 Comments and details

Factorization of $F_{q,q'}(\xi, \xi')$. Let us insert in Eq. (46) the Fourier representation (31) in the previous expression and change to the variables $R = \frac{1}{u}$, $R' = \frac{1}{u'}$:

$$F_{q,q'}(\xi, \xi') = \frac{\int \frac{du}{u} \frac{du'}{u'} \frac{dr}{r} d\kappa r^{i(q-q')} u^{-iq} u'^{iq'} e^{i\kappa \cdot (\eta - \eta')} e^{-\frac{u+u'}{2} \kappa^2 - \frac{(u+u')}{2uu'}} r^2}{\Gamma\left(\frac{d-2}{2} + iq\right) \Gamma\left(\frac{d-2}{2} - iq'\right)}. \quad (52)$$

⁴From (46) it is however seen that the case $d = 2$ exhibits some peculiarities since the term proportional to $\frac{1}{A(q)}$ does not vanish for $q = 0$.

The integral over κ can be factorized by the changes $r^2 = \sigma^2 u u' \kappa^2$, $u = \frac{v}{\kappa^2}$ and $u' = \frac{v'}{\kappa^2}$:

$$F_{q,q'}(\xi, \xi') = \frac{\Delta_{q,q'}}{\Gamma\left(\frac{d-2}{2} + iq\right) \Gamma\left(\frac{d-2}{2} - iq'\right)} \int d\kappa \kappa^{i(q-q')} e^{i\kappa \cdot (\eta - \eta')} \quad (53)$$

$$\Delta_{q,q'} = \int \frac{d\sigma}{\sigma} \sigma^{i(q-q')} \int \frac{dv}{v} v^{-i\frac{q+q'}{2}} e^{-\frac{v}{2} - \frac{v'}{2} \sigma^2} \int \frac{dv'}{v'} v'^{i\frac{q+q'}{2}} e^{-\frac{v'}{2} - \frac{v}{2} \sigma^2}. \quad (54)$$

Evaluation of $\Delta_{q,q'}$.

1) $q + q' \neq 0$. The second and third integrals at RHS can be evaluated as follows:

$$\int \frac{dv}{v} v^{-i\frac{q+q'}{2}} e^{-\frac{v}{2} - \frac{v'}{2} \sigma^2} = 2^{-\frac{iq+iq'}{2}} (1 + \sigma^2)^{\frac{iq+iq'}{2}} \Gamma\left(-\frac{iq+iq'}{2}\right). \quad (55)$$

The integral converges only if $\text{Im}(q + q') > 0$. Otherwise it is defined in a generalized sense as a meromorphic function of the complex $(q + q')$ variable. It then follows that

$$\Delta_{q,q'} = \Gamma(-iq) \Gamma(iq) 2\pi \delta(q - q'). \quad (56)$$

2) $q - q' \neq 0$. In this case we can extract a finite contribution by exchanging the integration order in Eq. (54):

$$\int \frac{d\sigma}{\sigma} \sigma^{i(q-q')} e^{-\frac{v+v'}{2} \sigma^2} = \frac{1}{2} \left(\frac{v+v'}{2}\right)^{-\frac{iq-iq'}{2}} \Gamma\left(\frac{iq-iq'}{2}\right). \quad (57)$$

This expression is valid for $\text{Im}(q - q') < 0$. Otherwise it is defined in a generalized sense as a meromorphic function of the complex $(q - q')$ variable. By introducing the variables $\lambda = \frac{v'}{v}$ and $\mu = \frac{v+v'}{2}$ Eq. (54) becomes

$$\Delta_{q,q'} = \frac{1}{2} \Gamma\left(\frac{iq-iq'}{2}\right) \int \frac{d\lambda}{\lambda} \frac{d\mu}{\mu} \mu^{-\frac{iq-iq'}{2}} \lambda^{\frac{iq+iq'}{2}} e^{-\mu} = \Gamma(iq) \Gamma(-iq) 2\pi \delta(q + q'). \quad (58)$$

Gathering all terms together finally yields

$$\Delta_{q,q'} = 2\pi \Gamma(iq) \Gamma(-iq) (\delta(q - q') + \delta(q + q')). \quad (59)$$

and Eq. (46) follows from Eq. (53).

6 Sketch of an integral transformation theory. Completeness.

6.1 Transform and inversion.

Let $f(x)$ be a smooth function defined on \mathbb{H}^{d-1} . We define the following transform [20]:

$$f(x) \rightarrow \tilde{f}(\xi, q) = \int d\mu(x) (x \cdot \xi)^{-\frac{d-2}{2} - iq} f(x). \quad (60)$$

Variables paired by the transform are

$$\mathbb{H}^{d-1} \ni x \longleftrightarrow (\xi, q) \in C^+ \times \mathbb{R}. \quad (61)$$

Since the transformed function is an homogeneous function of the ξ variable, it is enough to take ξ on the absolute and therefore there are $(d-1)$ dimensions on both sides of this pairing. The aim is now to reconstruct the function $f(x)$ in terms of $\tilde{f}(\xi, q)$ by inverting the previous transform. The inversion formula we will heuristically prove in this section is the following⁵:

$$f(x) = \int_0^\infty n(q) dq \int d\mu(\xi) (x \cdot \xi)^{-\frac{d-2}{2}+iq} \tilde{f}(\xi, q) \quad (62)$$

where $n(q)$ is given in Eq. (47)

6.2 Completeness.

Let us consider the (formal) integral operator

$$\hat{\delta}(x, x') = \int_0^\infty dq' \int d\mu(\xi) \psi_{iq'}(x, \xi) \psi_{iq'}^*(x', \xi) \quad (63)$$

in the Hilbert space $L^2(\mathbb{H}^{d-1}, d\mu)$. Using the coordinates (4), this Hilbert space may be concretely realized as a tensor product:

$$L^2(\mathbb{H}^{d-1}, d\mu) = L^2(\mathbb{R}^+ \times \mathbb{R}^{d-2}, r^{1-d} dr dx) = L^2(\mathbb{R}^+, r^{1-d} dr) \otimes L^2(\mathbb{R}^{d-2}, dx). \quad (64)$$

The space $L^2(\mathbb{H}^{d-1}, d\mu)$ is therefore generated by finite linear combinations of factorized functions $f(r, x) = g(r)h(x)$ where the factors are such that $g(r) \in L^2(\mathbb{R}^+, r^{1-d} dr)$ and $h \in L^2(\mathbb{R}^{d-2}, dx)$. Consider one function of this type. Using the integral representation (31) we can express the operator (63) as follows:

$$\begin{aligned} \int d\mu(x') \hat{\delta}(x, x') f(x') &= \frac{r^{\frac{d-2}{2}}}{(2\pi)^{d-2}} \int d\kappa \int dx' e^{i\kappa \cdot (x-x')} h(x') \\ &\times \int_0^\infty dq \frac{2}{\pi \Gamma(iq) \Gamma(-iq)} K_{iq}(\kappa r) \int_0^\infty \frac{dr'}{r'} K_{iq}(\kappa r') \frac{g(r')}{r'^{\frac{d-2}{2}}}. \end{aligned} \quad (65)$$

The function $r^{-\frac{d-2}{2}} g(r) \in L^2(\mathbb{R}^+, r^{-1} dr)$ and this assures convergence of the inner integral. The integral over r' and q are then just an instance of the Kontorovich-Lebedev [26] inversion formula, that holds true for quite general classes of functions and distributions:

$$\frac{g(r)}{r^{\frac{d-2}{2}}} = \int_0^\infty dq \frac{2}{\pi \Gamma(iq) \Gamma(-iq)} K_{iq}(\kappa r) \int_0^\infty \frac{dr'}{r'} K_{iq}(\kappa r') \frac{g(r')}{r'^{\frac{d-2}{2}}}. \quad (66)$$

The remaining integral in Eq. (65) is then just Fourier inversion formula. Taking finite linear combinations we finally get

$$\int d\mu(x') \hat{\delta}(x, x') f(x') = f(x) \quad (67)$$

on (a dense subset) of $L^2(\mathbb{H}^{d-1}, d\mu)$. This shows the validity of the inversion formula (62).

⁵Another choice would be to define a normalized transform:

$$f(x) \rightarrow \tilde{f}(\xi, q) = \int d\mu(x) \psi_{iq}^*(x, \xi) f(x).$$

In this case the inversion formula would have unit weight

$$f(x) = \int_0^\infty dq \int d\mu(\xi) \psi_{iq}(x, \xi) \tilde{f}(\xi, q).$$

Our above definition (60) follows similar choices of normalization such as the Mehler-Fock or the Kontorovich-Lebedev transforms, see e.g. [24], Eqs. (3.15; 8-9).

7 Projectors. Representations of the principal series.

At this point we may introduce the integral kernels

$$\Pi_q(x, x') = \int d\mu(\xi) \psi_{iq}(x, \xi) \psi_{iq}^*(x', \xi). \quad (68)$$

It is immediately seen that the kernels Π_q satisfy the following projector relations:

$$\int d\mu(x'') \Pi_q(x, x'') \Pi_{q'}(x'', x') = \delta(q - q') \Pi_q(x, x'). \quad (69)$$

The operator $\Pi_q(x, x')$ is the projector on the subspace of a given q^2 and as such $\Pi_q(x, x')$ is a positive-definite kernel. Starting from the projector $\Pi_q(x, x')$ we can construct a representation of the invariance group of \mathbb{H}^{d-1} in the usual way: let us consider the space of smooth rapidly decreasing functions on $\mathcal{S}(\mathbb{H}^{d-1})$ endowed with the left regular action $(T_g f)(x) = f(g^{-1}x)$, $g \in SO_0(1, d-1)$. As usual, let us introduce in such a space the scalar product

$$\langle f, f' \rangle = \int_{\mathbb{H}^{d-1}} f^*(x) \Pi_q(x, x') f'(x') dx dx' \quad (70)$$

$\mathcal{S}(\mathbb{H}^{d-1})$ is a pre-Hilbert space. By quotienting and completing we obtain a Hilbert space carrying an irreducible unitary representation of the Lorentz group labeled by the real, non-negative parameter q . The set of such representations is called the principal series.

Consider now any invariant (possibly positive-definite) two-point kernel $W(x, x')$ on the hyperboloid \mathbb{H}^{d-1} . If we assume suitable growth properties of W at infinity we may expect that it can be decomposed as a superposition of the projectors $\Pi_q(x, x')$ (representations of the principal series):

$$W(x, x') = \int_0^\infty \rho(q) \Pi_q(x, x') dq. \quad (71)$$

In particular the kernel $\hat{\delta}(x, x')$, as defined in Section (6.2), determines the standard L^2 Hilbert product on the hyperboloid \mathbb{H}^{d-1} and the so-called regular representation:

$$\hat{\delta}(x, x') = \int_0^\infty \Pi_q(x, x') dq \quad (72)$$

which can be equivalently viewed as the decomposition of the regular representation into representations of the principal series (Plancherel's formula):

$$\int d\mu(x) f^*(x) g(x) = \int_0^\infty n(q) dq \int d\mu(\xi) \tilde{f}^*(\xi, q) \tilde{g}(\xi, q). \quad (73)$$

Evaluation of $\Pi_q(x, x')$. The explicit evaluation of the integral (68) is most easily done by integrating on the spherical basis (18) of the absolute ($\xi^0 = 1$). The result must depend only on the scalar product $x \cdot x'$: without loss of generality we may choose $x = (1, 0, \dots, 0)$ and $x' = (\cosh \phi, -\sinh \phi, \dots, 0)$ so that $x \cdot x' = \cosh \phi$. Since $x \cdot \xi = 1$, and $x' \cdot \xi = \cosh \phi + \cos \theta_1 \sinh \phi$, the integral (68) becomes

$$\begin{aligned} & \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} n(q) \int_0^\pi d\theta_1 (\cosh \phi + \cos \theta_1 \sinh \phi)^{-\frac{d-2}{2}-iq} (\sin \theta_1)^{d-3} = \\ & = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)} n(q) \Gamma\left(\frac{d-1}{2}\right) 2^{\frac{d-3}{2}} (\sinh \phi)^{-\frac{d-3}{2}} P_{-\frac{1}{2}+iq}^{-\frac{d-3}{2}}(\cosh \phi) \end{aligned} \quad (74)$$

so that the final result reads

$$\Pi_q(x, x') = \omega_{d-1} n(q) P_{-\frac{d-2}{2}+iq}^{(d)}(x \cdot x'). \quad (75)$$

The factor ω_{d-1} is the hypersurface of the sphere \mathbb{S}^{d-2} (see Footnote 1) and $n(q)$ is given by (47). The result is expressed in terms of the so-called generalized Legendre function:

$$P_{-\frac{d-2}{2}+iq}^{(d)}(z) = 2^{\frac{d-3}{2}} \Gamma\left(\frac{d-1}{2}\right) (z^2 - 1)^{-\frac{d-3}{4}} P_{-\frac{1}{2}+iq}^{-\frac{d-3}{2}}(z) \quad (76)$$

where $P_\nu^\mu(z)$ denotes the usual Legendre function of the first kind, defined and one-valued in the complex z -plane cut on the reals from $-\infty$ to 1 [24]. The function $(z^2 - 1)^\alpha$ is defined and one valued on the same cut complex plane (with the natural definition for real $z > 1$) so that the function $P_{-\frac{d-2}{2}+iq}^{(d)}(z)$ is regular at $z = 1$ and its cut goes from $z = -\infty$ to $z = -1$ only.

8 Milne's universe.

As a first application of the general construction displayed in the previous sections, we discuss here Quantum Field Theory in the universe of Milne.

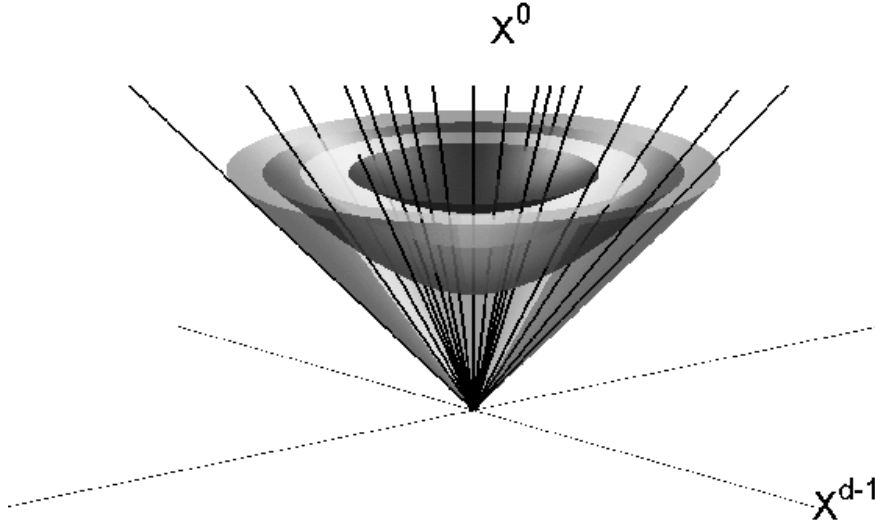


Figure 5: *A view of the Milne universe. The spacetime curvature of this model is zero. Surfaces of constant time are copies of the Lobatchevski space \mathbb{H}^{d-1} . The straight lines represented in figure are the geodesic worldlines of particles having constant spatial coordinates.*

Milne's universe is a simple model of an expanding universe obtained as a solution of the Einstein equations in vacuo with zero spacetime curvature and nonzero spatial curvature. There is however no need of General Relativity to talk about this model: Fig. 5 shows how the Milne universe can be constructed as a foliation with Lobatchevskian leaves of the interior of the the future cone of an event (the "Big Bang") of a Minkowski spacetime. A quantitative description is very easy: let X^μ denote

the coordinates of an event X of a (d -dimensional) Minkowski spacetime. Consider the future cone of the origin of the chosen inertial system (as in Fig. 5) and introduce there the noninertial coordinate system:

$$X^\mu(t, x) = t x^\mu, \quad \mu = 0, \dots, d-1, \quad (77)$$

where $x \cdot x = 1$, i.e. $x \in \mathbb{H}^{d-1}$. Milne's line element is simply Lorentz invariant interval of the ambient spacetime expressed in the coordinates (77):

$$ds^2 = (dX^0)^2 - (dX^1)^2 - \dots - (dX^{d-1})^2 = dt^2 - t^2 dl_{d-1}^2. \quad (78)$$

Milne's universe has therefore the structure of a warped product of a half line (the cosmic time) times the Riemannian manifold \mathbb{H}^{d-1} ; the warping function is just the cosmic time t (see e.g [21]).

As old as it is, this model has never become obsolete and disappeared from the scientific and cosmological debate; its predictions are in surprisingly good agreement with the current cosmological observations [23]. A recent appearance of Milne's model in M-theory is also worth to be mentioned [22].

The simple question we ask ourselves here is that of finding the expansion of the exponential plane waves of the Minkowski spacetime on the base of modes (48). This is a preliminary step to study quantum theories on the Milne universe in much the same way as finding an expansion of plane waves in spherical harmonics is a starting point in studying spherical symmetric potentials in ordinary quantum mechanics. As an immediate bonus of our approach there is an easy construction the Wightman vacuum in Milne's coordinates. We give here a new and we believe simple approach to solve this old problem [27]. Our approach may also be used to study other vacua, as for instance the thermal vacuum at a given temperature.

8.1 Free fields

To fix ideas and notations, and also to put the results in perspective, it is useful to begin by shortly reviewing the theory of a Klein-Gordon quantum field of mass m on a d -dimensional Minkowski spacetime \mathbb{M}^d with inertial coordinates $(X^0, X^1, \dots, X^{d-1})$:

$$(\square + m^2)\phi = 0. \quad (79)$$

It is enough to solve the Klein-Gordon equation for the two-point vacuum expectation value

$$W_m(X - Y) = \langle \Omega, \phi(X)\phi(Y)\Omega \rangle. \quad (80)$$

The truncated n -point functions are assumed to vanish and the two-point function encodes all the information necessary to fully reconstruct the theory. Actually, for Klein-Gordon fields satisfying the Wightman axioms the vanishing of the truncated n -point functions is not an assumption but it is a result [28] and a Klein-Gordon field is necessarily free. Eq. (79) is most easily solved in Fourier space, where it becomes algebraic:

$$(p^2 - m^2)\tilde{W}_m(p) = 0. \quad (81)$$

There are infinitely many inequivalent solutions and a criterium is to be found to select one among them. Assumption of positivity of the spectrum of the energy-momentum operator is the most popular possibility and leads to

$$\tilde{W}_m(p) \simeq \theta(p^0)\delta(p^2 - m^2), \quad (82)$$

where $\theta(p^0)$ denotes Heaviside's step function. Inversion gives

$$W_m(X, X') = \langle \Omega, \phi(X)\phi(X')\Omega \rangle = \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} dp e^{-ip \cdot (X - X')} \theta(p^0)\delta(p^2 - m^2). \quad (83)$$

The choice made in Eq. (82) selects the Wightman vacuum Ω which is uniquely characterized by the positivity of the spectrum of the energy operator in any Lorentz frame. This property is equivalent to

certain analyticity properties of the correlation functions that can be deduced by direct inspection of Eq. (83). One sees that the Wightman function W can be uniquely extended to a function holomorphic in the past tube T^- as a function of the difference variable $(X - X')$ where

$$T^- = \{X + iY, \ Y^2 > 0, Y_0 < 0\}. \quad (84)$$

If we consider the plane waves on the mass shell we see that positive frequency waves $\exp(-i\sqrt{p^2 + m^2}X^0 + i\vec{p} \cdot \vec{X})$ admit a natural continuation in the past tube where they are decreasing while negative frequency waves $\exp(i\sqrt{p^2 + m^2}X^0 - i\vec{p} \cdot \vec{X})$ may be considered for complex events belonging to the future tube

$$T^+ = \{X + iY, \ Y^2 > 0, Y^0 > 0\}. \quad (85)$$

These properties are the link between the standard choices in the canonical quantization procedure and the analyticity structure of the waves and the Wightman function.

8.2 Plane waves: projection and inversion. QFT.

Let us consider therefore a Minkowskian plane wave on the mass shell $p^2 = m^2$, $p^0 > 0$ written in Milne's coordinates:

$$\exp(ip \cdot X) = \exp(it p \cdot x). \quad (86)$$

The wave is naturally extended in the future tube T^+ ; in particular we will consider the complex events

$$Z^\mu(\tau, x) = \tau x^\mu, \quad \tau = t + is, \quad \text{Im } \tau = s > 0, \quad (87)$$

that belong to the future tube. Similarly the wave $\exp(-i\tau p \cdot x)$ is naturally extended to the past tube and in particular we will consider the events $Z^\mu(\tau, x) \in T^-$ that belong in the past tube for $\text{Im } \tau < 0$.

We now look for an expansion of the Minkowskian plane waves adapted to Milne's geometry. As explained in the previous sections, the first step is to compute their scalar products with the modes (24) as follows:

$$F_q^\pm(\tau, \xi, p) \int_{\mathbb{H}^{d-1}} d\mu(x) (x \cdot \xi)^{-\frac{d-2}{2}-iq} e^{\pm i\tau p \cdot x}. \quad (88)$$

Here F_q^\pm are defined respectively for $X(\tau, x) \in T^\pm$, i.e. $\text{Im } \tau > 0$ resp. $\text{Im } \tau < 0$. The Lorentz invariance of the measure implies that F_q^\pm may depend only on the invariant $(\xi \cdot p)$. Homogeneity of the integrand then gives that

$$F_q^\pm(\tau, \xi, p) = f_q^\pm(\tau) (\xi \cdot p)^{-\frac{d-2}{2}-iq}.$$

The steps to explicitly compute the function f are summarized at the end of this section; here is the result:

$$F_q^+(\tau', \xi, p) = i\pi \left(\frac{2\pi i}{m\tau'} \right)^{\frac{d-2}{2}} \left(\frac{p \cdot \xi}{m} \right)^{-\frac{d-2}{2}-iq} e^{-\pi q H_{iq}^{(1)}(m\tau')}, \quad \text{Im } \tau' > 0 \quad (89)$$

$$F_q^-(\tau, \xi, p) = \frac{\pi}{i} \left(\frac{2\pi}{im\tau} \right)^{\frac{d-2}{2}} \left(\frac{p \cdot \xi}{m} \right)^{-\frac{d-2}{2}-iq} e^{\pi q H_{iq}^{(2)}(m\tau)}, \quad \text{Im } \tau < 0 \quad (90)$$

Note that $H_{iq}^{(2)}(m\tau) \propto e^{-im\tau}$ when $\tau \rightarrow \infty$. As expected [27] the Hankel function $H_{iq}^{(2)}$ plays the role of the positive frequency solution of the Klein-Gordon equation when separated in the coordinates (77). The result in our construction comes out automatically from the known analyticity properties (84) of the Minkowskian waves.

Inversion is obtained by means of Eq. (62); this yields the expansion of the exponential plane wave (86) in terms of the wavefunctions (24); the $(d-1)$ parameters p are described by the $(d-2)$ degrees of freedom of ξ on the absolute plus one degree of freedom of the q variable:

$$\begin{aligned} \exp i\tau(p \cdot x) &= i\pi \left(\frac{2\pi i}{m\tau} \right)^{\frac{d-2}{2}} \int_0^\infty n(q) dq e^{-\pi q} H_{iq}^{(1)}(m\tau) \\ &\quad \times \int d\mu(\xi) \left(\frac{p \cdot \xi}{m} \right)^{-\frac{d-2}{2}-iq} (x \cdot \xi)^{-\frac{d-2}{2}+iq} \end{aligned} \quad (91)$$

and similarly for the other wave. The integration over the absolute at the RHS can be performed (see Eqs. (68) and (75)) and there results a one-dimensional integral expansion over the projectors Π_q as follows:

$$\begin{aligned} e^{i\tau(p \cdot x)} &= i\pi \left(\frac{2\pi i}{m\tau} \right)^{\frac{d-2}{2}} \int_0^\infty dq e^{-\pi q} H_{iq}^{(1)}(m\tau) \Pi_q \left(\frac{p \cdot x}{m} \right), \quad \text{Im } \tau > 0, \\ e^{-i\tau(p \cdot x)} &= -i\pi \left(\frac{2\pi}{im\tau} \right)^{\frac{d-2}{2}} \int_0^\infty dq e^{\pi q} H_{iq}^{(2)}(m\tau) \Pi_q \left(\frac{p \cdot x}{m} \right), \quad \text{Im } \tau < 0. \end{aligned} \quad (92)$$

The details of the calculation are given at the end of the present section. As an immediate bonus these formulae provide an expansion of the Wightman canonical Klein-Gordon quantum field theory expressed in Milne's coordinates. Indeed, since the theory is completely encoded in the two-point function and the invariant measure on the mass shell is proportional to the invariant measure $d\mu$ on \mathbb{H}^{d-1} , we can insert Eqs. (92) into Eq. (83) and use Eq. (69) to get

$$\begin{aligned} W_m(X, X') &= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{R}^d} dp e^{-ip \cdot (X-X')} \theta(p_0) \delta(p^2 - m^2) \\ &= \frac{1}{4} \pi (\tau\tau')^{-\frac{d-2}{2}} \int dq H_{iq}^{(2)}(m\tau) H_{iq}^{(1)}(m\tau') \Pi_q(x \cdot x'), \end{aligned} \quad (93)$$

where $\text{Im } \tau < 0, \text{Im } \tau' > 0$. As a verification, let us show that the theory is indeed canonical by computing the following equal time commutation relations:

$$\begin{aligned} [\phi(t, x), \pi(t, x')] &= \\ &= \frac{\pi m t}{4} \int dq \left(H_{iq}^{(2)}(mt) H_{iq}^{(1)'}(mt) - H_{iq}^{(2)'}(mt) H_{iq}^{(1)}(mt) \right) \Pi_q(x \cdot x') \\ &= i \delta(x, x'), \end{aligned} \quad (94)$$

where $\delta(x, x')$ is understood in the sense of Section (6.2).

Comments and details

Evaluation of F_q . In this section we show that the expressions of F_q as given in Eqs. (89) and (90) hold true. To this purpose, let us parametrize x as in Eq. (4) and similarly write the momentum vector p as follows:

$$p = \begin{cases} p^0 &= \frac{m}{2\lambda}(1 + \kappa^2 + \lambda^2) \\ p^i &= \frac{m}{2\lambda}\kappa^i \\ p^{d-1} &= \frac{m}{2\lambda}(1 - \kappa^2 - \lambda^2) \end{cases}. \quad (95)$$

This yields

$$\exp ip \cdot X = \exp i\tau p \cdot x = \exp \frac{im\tau}{2r\lambda} [(x - \kappa)^2 + r^2 + \lambda^2]. \quad (96)$$

By using the integral representation (30) and performing the Gaussian integral one gets

$$F_q^+ = \frac{(2\pi)^{\frac{d-2}{2}} i^{-\frac{d-2}{2}-iq}}{\Gamma\left(\frac{d-2}{2} + iq\right)} \int \frac{dR}{R} \frac{R^{\frac{d-2}{2}+iq}}{[-i(T+R)]^{\frac{d-2}{2}}} \int \frac{dr}{r} r^{-\frac{d-2}{2}} e^{\frac{ir(R+T)}{2} + \frac{i\lambda^2 T}{2r} + i\frac{TR(\kappa-\eta)^2}{2r(T+R)}} \quad (97)$$

where the integral over R is along an arbitrary straight half-line such that $0 < \text{Arg}(R) < \pi$; to simplify notations we have put

$$T = \frac{m}{\lambda}(t + is) = \frac{m\tau}{\lambda}. \quad (98)$$

The evaluation of the remaining integrals is simplified by the introduction of a new complex variable replacing the r -coordinate: given R and T in the upper complex plane we define

$$v = \left(\frac{1}{R} + \frac{1}{T} \right) r, \quad -\pi < \text{Arg}(v) < 0. \quad (99)$$

Let us use the freedom in choosing the integration path in the complex R -plane and take $\text{Arg}(R) = \text{Arg}(T)$ (path γ); it follows that $\text{Arg}(v) = -\text{Arg}(R)$ (path $\hat{\gamma}$) and the previous expression becomes

$$F_q^+ = \frac{i^{-\frac{d-2}{2}-iq}}{\Gamma(\frac{d-2}{2}+iq)} \left(\frac{2i\pi}{T} \right)^{\frac{d-2}{2}} \int_{\gamma} \frac{dR}{R} R^{iq} \int_{\hat{\gamma}} \frac{dv}{v} v^{-\frac{d-2}{2}} e^{\frac{iRTv}{2} + \frac{i\lambda^2(R+T)}{2Rv}} e^{\frac{i(\kappa-\eta)^2}{2v}}. \quad (100)$$

In this expression we interchange the integration order, introduce the real variable $S = Rv$ and the inversion $u = \frac{\lambda}{v}$ (that implies $0 < \text{Arg}(u) = \text{Arg}(R) < \pi$):

$$F_q^+ = \frac{(i\lambda)^{-\frac{d-2}{2}-iq}}{\Gamma(\frac{d-2}{2}+iq)} \left(\frac{2i\pi}{T} \right)^{\frac{d-2}{2}} \int_{\gamma} \frac{du}{u} u^{\frac{d-2}{2}+iq} e^{\frac{i(\kappa-\eta)^2 + i\lambda^2}{2\lambda} u} \int_0^\infty \frac{dS}{S} S^{iq} e^{\frac{iST}{2} + \frac{i\lambda^2 T}{2S}}. \quad (101)$$

In the previous expression we recognize (see Eq. (30)) the scalar

$$\left(\frac{p \cdot \xi}{m} \right)^{-\frac{d-2}{2}-iq} = \frac{i^{-\frac{d-2}{2}-iq}}{\Gamma(\frac{d-2}{2}+iq)} \int_{\gamma} \frac{du}{u} u^{\frac{d-2}{2}+iq} e^{iu \left[\frac{(\kappa-\eta)^2}{2\lambda} + \frac{\lambda}{2} \right]} \quad (102)$$

and the representation (32) of the Bessel function $K_{iq}(z)$. Taking into account Eq. (98) and the phase of τ we get

$$\begin{aligned} F_q^+(\tau, \xi, p) &= 2 \left(\frac{2\pi i}{m\tau} \right)^{\frac{d-2}{2}} \left(\frac{p \cdot \xi}{m} \right)^{-\frac{d-2}{2}-iq} K_{iq}(-im\tau) \\ &= i\pi \left(\frac{2\pi i}{m\tau} \right)^{\frac{d-2}{2}} \left(\frac{p \cdot \xi}{m} \right)^{-\frac{d-2}{2}-iq} e^{-\pi q} H_{iq}^{(1)}(m\tau), \quad \text{Im } \tau > 0. \end{aligned} \quad (103)$$

This is the result given in Eq. (89). The expression (90) is obtained by complex conjugation, as can be readily inferred from (88). This completes the proof of the expansions given in Eqs. (92).

9 QFT on the open de Sitter universe

9.1 General considerations. Geometry.

In this section we will apply our construction to the open de Sitter universe. The simplest possible description of this model is as follows: consider a $(d+1)$ -dimensional Minkowski spacetime with inner product

$$X \cdot Y = X^0 Y^0 - X^1 Y^1 - \dots - X^d Y^d, \quad (104)$$

and the embedded Lorentzian d -dimensional manifold dS_d with equation

$$dS_d = \{X : X^2 = X \cdot X = -1\}; \quad (105)$$

this manifold models the whole de Sitter universe (the grey manifold in Fig. 6). Let us consider now the intersection of dS_d with the future cone V_+ of a given event. Specifically, we consider the future region of the event $O = (0, \dots, 0, 1)$ (the "origin"):

$$\Gamma^+ = \Gamma_O^+ = \{O + V^+\} \cap dS_d = \{X \in dS_d : (X - O)^2 > 0, X^0 > 0\}. \quad (106)$$

Γ^+ can be thought as a warped manifold, foliated with $(d-1)$ -dimensional Lobatchevskian leaves \mathbb{H}^{d-1} . Precisely, we have the following construction:

$$X(t, x) = \begin{cases} X^i &= \sinh t \, x^i, \quad i = 0, \dots, d-1 \\ X^d &= \cosh t \end{cases} \quad (107)$$

where $t > 0$, and $x^2 = 1$, i.e. $x \in \mathbb{H}^{d-1}$. In these coordinates the de Sitter metric is written as follows:

$$ds^2 = \left\{ (dX^0)^2 - (dX^1)^2 - \dots - (dX^d)^2 \right\} \Big|_{dS_d} = dt^2 - \sinh^2 t \, dl_{d-1}^2; \quad (108)$$

this is a warped product with warping function $a(t) = \sinh t$; in cosmology such a metric defines a particular instance of a Friedmann-Robertson-Walker hyperbolic universe. We will call the region Γ^+ parametrized with the cosmic time given chosen in (107) the *open de Sitter universe* (Fig. 6).

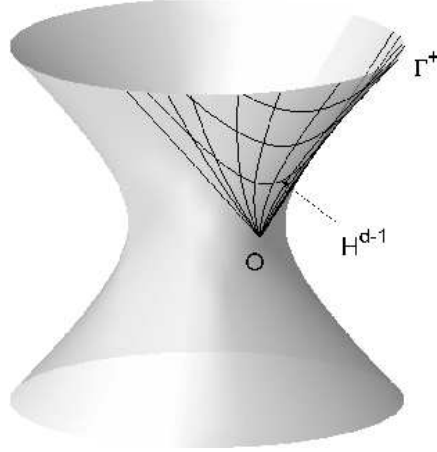


Figure 6: *The open de Sitter model is the interior of the future cone Γ_X^+ of any given event on the de Sitter manifold (in the figure $X=O$). The surfaces of constant time are copies of the Lobatchevski space \mathbb{H}^{d-1} . The geodesic worldlines of particles having constant spatial coordinates are branches of hyperbolae.*

As briefly mentioned in the Introduction, the open de Sitter universe was popular in the mid-nineties when it was playing a central role in open models of inflation [5, 7, 8]. These models were abandoned when the microwave background fluctuation measurement showed that our universe is most likely flat. From both the physical and mathematical viewpoints, correctly quantizing a field in the open de Sitter manifold is an arduous task. In particular, a naive application of the procedures of canonical quantization gives a "wrong" result [9, 13] in the sense that when those procedures are applied to the open de Sitter manifold one does not end up with the standard de Sitter invariant vacuum [1, 10, 12, 30]. One reason is that the spatial manifold \mathbb{H}^{d-1} represented in Fig. 6 is a complete Cauchy surface for Γ^+ that is its own Cauchy development (i.e. its future), but it fails to be so for the whole de Sitter manifold where it cannot be used to set up the usual canonical quantization. In [9] this difficulty was circumvented by finding an extension of the modes, originally defined only in the physical region Γ^+ , to the whole de Sitter manifold and applying the canonical quantization there. The major drawback of this approach is that it is strictly

limited to the de Sitter geometry. The method used in [13] was not based on canonical quantization but also necessitated the extension of the open de Sitter manifold to the whole manifold. These calculations thus are flawed by the necessity, in order to explain local events, to use (possibly complex) extensions of space-time to classically unreachable regions, whose a priori existence or non-existence in more general situation cannot be simply established. This drawback is avoided in the present approach since by choice we work solely in the physical space, here the open de Sitter manifold.

In the following we will perform in the open de Sitter universe the same Fourier-type analysis already described in the Milne's case. As before, an immediate bonus is the complete resolution of the standard de Sitter QFT solely in terms of the modes of the physical region Γ^+ . One valuable aspect of the method that we use is that it can also in principle be used to analyze observational data. Our discussion is limited here to the square-integrable case. Theories that involve modes that are not square-integrable case as well as the implications of our analysis for general canonical quantum theory will be the object of separate studies.

9.2 The complex de Sitter hyperboloid

In the study of quantum field theory on a given background, the complexification of the underlying manifold plays a central role in either the study of general properties (as for instance the PCT theorem) or in the construction of concrete models, which are essentially based on Euclidean methods, i.e. on the analytical continuation in the time coordinate. The de Sitter case is no exception to this rule.

The complex de Sitter spacetime can be described as a complex manifold embedded in the $(d+1)$ -dimensional complex Minkowski spacetime with equation:

$$dS_d^{(c)} = \{Z = X + iY \in \mathbb{M}^{(c)} : Z \cdot Z = -1\} \quad (109)$$

As in the flat case (see Eqs. 84 and 85) we introduce the tubular domains \mathcal{T}^+ and \mathcal{T}^- :

$$\mathcal{T}_d^+ = \{Z \in dS_d^{(c)} : Y \cdot Y > 0; Y^0 > 0\}, \quad (110)$$

$$\mathcal{T}_d^- = \{Z \in dS_d^{(c)} : Y \cdot Y > 0; Y^0 < 0\}, \quad (111)$$

defined as the intersection of the complex Sitter manifold with the the forward and backward tubes in the ambient complex Minkowski spacetime. These domains arise in connection with the thermal physical interpretation of de Sitter QFT [30, 31]. More precisely, assumption of analyticity of the correlation functions in (generalizations of) these domains give rise to the KMS property and therefore to the thermal interpretation: a de Sitter geodesic observer perceive a thermal bath of "particles".

As in the Milne case, special attention will be devoted to the complex events

$$Z(\tau, x) = \begin{cases} Z^i &= \sinh \tau \, x^i, \quad i = 0, \dots, d-1 \\ Z^d &= \cosh \tau \end{cases} \quad (112)$$

where only the cosmic time has been complexified. The complex coordinate $\tau = t + is$ is defined in the strip

$$\text{Im } \tau = s \in (-\pi, \pi) \quad (113)$$

of the complex τ -plane. Events $Z(\tau, x)$ such that $\text{Im } \tau \in (0, \pi)$ belong to \mathcal{T}^+ ; events $Z(\tau, x)$ such that $\text{Im } \tau \in (-\pi, 0)$ belong to \mathcal{T}^- .

An alternative description of these events can be given by using the variable $u = Z^d = \cosh \tau$. The image of both \mathcal{T}^+ and \mathcal{T}^- in the u -variable is the cut plane

$$\Delta = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\} \quad (114)$$

We are then led to consider the mappings $u \rightarrow Z^\pm(u, x)$ defined in $\Delta \times \mathbb{H}^{d-1}$ as follows⁶:

$$Z^\pm(u, x) = \begin{cases} Z^l &= \pm i(1-u^2)^{\frac{1}{2}} x^l, \quad l = 0, \dots, d-1 \\ Z^d &= u \end{cases} \quad (115)$$

It is readily seen that $Z^+(u, x) \in \mathcal{T}^+$ and $Z^-(u, x) \in \mathcal{T}^-$. Indeed, consider for instance the mapping

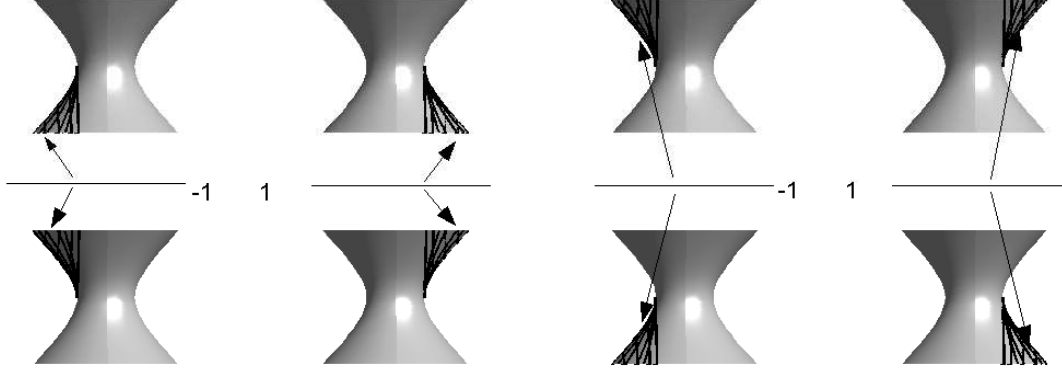


Figure 7: Two copies (left and right of the figure) of the cut plane Δ of the complex u variable showing the cuts at $(-\infty, -1) \cup (1, +\infty)$. The images of the de Sitter manifold correspond to u infinitesimally close to the cuts, at the place where the manifolds are drawn. **Left:** points of Δ are mapped through $Z^-(u, x)$ into \mathcal{T}^- . The imaginary part of these points is contained in V^- (not represented). The copy at the lower right is the one which corresponds to the open de Sitter space that we solely consider in the present paper. **Right:** points of Δ are mapped through $Z^+(u, x)$ into \mathcal{T}^+ . The imaginary part of these points is contained in V^+ (not represented). The copy at the upper right is the one which corresponds to the open de Sitter space that we solely consider in the present paper.

$u \rightarrow Z^+(u, x)$. When u is real and such that $-1 < u < 1$, the events $Z^+(u, x)$ evidently belong to the future tube \mathcal{T}^+ because for such events $\text{Im } Z^+ \cdot \text{Im } Z^+ > 0$ and $\text{Im } Z^0 = x^0 \text{Im } (i\sqrt{1-u^2}) > 0$. Now, the first of these conditions holds true for any $u \in \Delta$. On the other hand, since the zeros of $\text{Im } (i\sqrt{1-u^2})$ all belong to the cuts of Δ one also has that

$$\text{Im } Z^+(u, x) = x^0 \text{Im } (i\sqrt{1-u^2}) > 0 \quad \text{for all } u \in \Delta, \quad (116)$$

and the result follows. Note that we have $Z^-(u, x) = [Z^+(u, x)]^*$ (see App. (A), Footnote (7)).

9.3 Plane waves.

Let us consider an eigenfunction ϕ of the de Sitter Klein-Gordon operator:

$$\square \phi + m^2 \phi = 0 \quad (117)$$

here \square denotes the de Sitter-d'Alembert operator (i.e. the Laplace-Beltrami operator relative to the de Sitter metric). The usual approach to such an equation in curved spacetimes consists in trying to solve it by separating the variables in a suitably chosen coordinate systems; here in the system (107).

⁶The change of complex variables $\tau \rightarrow u$ maps the strip $\text{Im } \tau \in (-\pi, \pi)$ onto a two sheeted manifold defined by the cuts of Δ . With this mapping we could have considered a function $Z(u, x)$ that coincides with $Z^+(u, x)$ on a sheet and with $Z^-(u, x)$ on the other sheet. Since the de Sitter global waves are defined either in \mathcal{T}^+ or in \mathcal{T}^- it is simpler to use the (one-sheeted) cut-plane Δ and two different functions mapping Δ to \mathcal{T}^+ and to \mathcal{T}^- . We use this viewpoint throughout the present section.

If we do that and factorize the wave ϕ by separating the variables according with the reference system (107)

$$\phi(X) = f(t) \psi_{iq}(x, \xi) \quad (118)$$

the time-dependent factor $f(t)$ is required to satisfy the equation:

$$\frac{1}{(\sinh t)^{d-1}} \frac{\partial}{\partial t} (\sinh t)^{d-1} \frac{\partial f}{\partial t} + \frac{1}{(\sinh t)^2} \left[\left(\frac{d-2}{2} \right)^2 + q^2 \right] f + m^2 f = 0. \quad (119)$$

There is also the possibility to introduce global waves in a coordinate - independent way [12, 30] by using the embedding of the de Sitter hyperboloid in the Minkowski ambient spacetime. Their construction is identical to that of the spatial wavefunctions (24) with an important difference: they are singular on $(d-1)$ -dimensional light-like submanifolds of dS_d . This difficulty can be overcome by moving to the complexification of the de Sitter spacetime (109). The physically relevant global waves can be defined as the functions

$$\text{Const} (Z \cdot \Xi)^{-\frac{d-1}{2} + i\nu} \quad (120)$$

where, as before, $\Xi = (\Xi^0, \dots, \Xi^d)$ belong to future lightcone in the ambient spacetime, i.e. it is a future directed null vector of the ambient space ($\Xi \cdot \Xi = 0$ and $\Xi^0 > 0$). The parameter ν is a complex number. The physical values it may take are real, or purely imaginary with $|\nu| \leq \frac{d-1}{2}$, and correspond to

$$m^2 = \left(\frac{d-1}{2} \right)^2 + \nu^2 \geq 0. \quad (121)$$

The waves (120) are analytic for Z in the tubular domains \mathcal{T}^+ or \mathcal{T}^- of $dS_d^{(c)}$, defined in (111). These analyticity properties are the counterpart in the de Sitter universe of the *spectral condition* of the Minkowski case [30, 31].

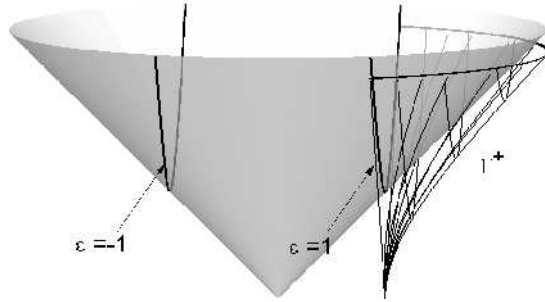


Figure 8: *The most convenient choice to represent the absolute of the open de Sitter space (shown at the right) is by a two-component manifold, labeled here by the two values of $\epsilon = \pm 1$. Each component is a copy of the Lobatchevski space \mathbb{H}^{d-1} , similar to the mass shell of the Minkowski case.*

Given the $(d+1)$ -dimensional vector $\Xi = (\Xi^0, \dots, \Xi^d)$ as above the d -dimensional vector $(\Xi^0, \dots, \Xi^{d-1})$ is timelike and forward directed. One has that $(\Xi^0, \dots, \Xi^{d-1}) = |\Xi^d|(a^0, \dots, a^{d-1})$ where $a^2 = 1$. There is no loss of generality in setting $|\Xi^d| = 1$; this is indeed another possible choice to represent the absolute of the ambient spacetime, and indeed the most convenient for our purposes. This manifold has two disconnected components:

$$\Xi \rightarrow (a, \epsilon), \quad \Xi = (a^0, \dots, a^{d-1}, \epsilon), \quad \epsilon = \pm 1 \quad (122)$$

From (112), we see that the scalar product

$$Z(\tau, x) \cdot \Xi(a, \epsilon) = x \cdot a \sinh \tau - \epsilon \cosh \tau, \quad (123)$$

has an imaginary part which does not vanish for events strictly within \mathcal{T}^+ or within \mathcal{T}^- . The waves (120) are therefore globally (but separately) well-defined in both these domains.

Representation of the plane waves.

The embedding of the de Sitter hyperboloid in the Minkowski ambient spacetime is the foundation of the construction of the global waves. The same embedding leads naturally [21, 19] to an integral representation of the de Sitter waves (120) in terms of the Minkowski coordinates. With a convenient choice of phase and normalization, the waves (120) may be expressed in the forward tube as follows:

$$(-iZ \cdot \Xi)^{-\frac{d-1}{2}+i\nu} = \frac{1}{\Gamma\left(\frac{d-1}{2}-i\nu\right)} \int_0^\infty \frac{dR}{R} R^{\frac{d-1}{2}-i\nu} e^{iR(Z \cdot \Xi)}, \quad Z \in \mathcal{T}^+. \quad (124)$$

The strictly positive imaginary part of $Z \cdot \Xi$ guarantees the proper definition of the wave as well (see above) as well as the convergence of the integral at infinity. The integral converges also at the origin provided $|\operatorname{Im} \nu| < \frac{d-1}{2}$.

A similar representation holds in \mathcal{T}^- :

$$(iZ \cdot \Xi)^{-\frac{d-1}{2}+i\nu} = \frac{1}{\Gamma\left(\frac{d-1}{2}-i\nu\right)} \int_0^\infty \frac{dR}{R} R^{\frac{d-1}{2}-i\nu} e^{-iR(Z \cdot \Xi)}, \quad Z \in \mathcal{T}^-. \quad (125)$$

Now it is the strictly negative imaginary part of $Z \cdot \Xi$ that makes the integral converge.

9.4 Analysis of plane waves in the open de Sitter universe.

In the following we want to provide an expansion of de Sitter waves in terms of the eigenmodes of the hyperbolic Laplacian. As before the relevant expansion can be obtained by computing the following integral transform

$$F_{q,\nu}^\pm(\tau, \xi, a, \epsilon) = \int_{\mathbb{H}^{d-1}} d\mu(x) (x \cdot \xi)^{-\frac{d-2}{2}-iq} (\mp iZ(\tau, x) \cdot \Xi(a, \epsilon))^{-\frac{d-1}{2}+i\nu}. \quad (126)$$

The integral is well-defined both for $0 < \operatorname{Im} \tau < \pi$ (corresponding to F^+) and $-\pi < \operatorname{Im} \tau < 0$ (corresponding to F^-) so as to avoid the vanishing of $X(\tau, x) \cdot \Xi(a, \epsilon)$. A glance to the large $x(r, \mathbf{x})$ behavior of (126) that corresponds to $r \rightarrow 0$, shows that the simultaneous convergence of $F_{q,\nu}^\pm$ and $F_{q,-\nu}^\pm$ is guaranteed by the condition $|\operatorname{Im} \nu| < \frac{1}{2}$, since we consider only real values of q . This condition sets a lower bound of the masses of the field theories that are covered by the present treatment.

As before $SO_0(1, d-1)$ invariance tells us that result must be function of the scalar product $\xi \cdot a$. Homogeneity then implies that

$$F_{q,\nu}^\pm(\tau, \xi, a, \epsilon) = f_{q,\nu}^\pm(\tau, \epsilon) (\xi \cdot a)^{-\frac{d-2}{2}-iq}. \quad (127)$$

The functions $f_{q,\nu}^\pm(t, \epsilon)$ are therefore the relevant solution of equation (119) that agree with the spectral condition described above.

9.5 Two-dimensional case.

We start by discussing the two-dimensional case, which deserves special consideration because of its simplicity. The spatial manifold \mathbb{H}^1 is one-dimensional and can be parametrized by an hyperbolic angle v :

$$x = (\cosh v, \sinh v), \quad d\mu(x) = dv \quad (128)$$

(i.e. $r = e^{-v}$ in Eq. (4)). Labeling the modes of the Laplacian is also quite simple: the spatial absolute has only two possible direction; consequently, the spatial momentum vector ξ can take only two discrete values: $\xi_l = (1, -1)$ and $\xi_r = (1, 1)$ so that $\xi_l \cdot x = \cosh v + \sinh v = e^v$ and $\xi_r \cdot x = e^{-v}$.

As regards the plane waves, they are labeled as in Eq. (122) by the discrete variable $\epsilon = \pm 1$ and by the one-dimensional vector a ; the latter may in turn be also parameterized by an hyperbolic angle $a = (\cosh w, \sinh w)$ so that

$$x \cdot a = \cosh(v - w)$$

The computation of the scalar product is then straightforward

$$\begin{aligned} F_{q,\nu}^{\pm}(\tau, \xi_l, a, \epsilon) &= f_{q,\nu}^{\pm}(\tau, \epsilon)(\xi_l \cdot a)^{-iq}, & F_{q,\nu}^{\pm}(\tau, \xi_r, a, \epsilon) &= f_{q,\nu}^{\pm}(\tau, \epsilon)(\xi_r \cdot a)^{-iq}, \\ f_{q,\nu}^{\pm}(\tau, \epsilon) &= \int_{-\infty}^{+\infty} dw e^{-iqw} (\mp i \cosh w \sinh \tau \pm i \epsilon \cosh \tau)^{-\frac{1}{2}+i\nu}. \end{aligned} \quad (129)$$

The two-dimensional case also provides an easy direct check of the inversion formula. Indeed, with the above choice of coordinates, it simply amounts to Fourier inversion.

Expression of f in terms of Legendre functions. The functions $f_{q,\nu}^{\pm}(\tau, \epsilon)$ are completely characterized by the integral representation (129). They can be however expressed in terms of the associated Legendre functions. We use here and in the following the notations and the conventions of the Bateman manuscript project [24] with one notable exception (the function \mathbf{Q} , see below). In particular the Legendre functions P and Q are assumed to be analytic and one valued on the complex plane cut from $z = -\infty$ to $z = 1$. The two cases to be considered look however at first rather different, since the integrand never vanishes for $\epsilon = -1$, while it become singular along the integration path for $\epsilon = 1$ (for τ real: $\tau = t > 0$).

Case $\epsilon = -1$. In this case the integral representation (129) already coincides with a well-known integral representation of a Legendre functions of the second kind [24] Eq. (3.7;12):

$$f_{q,\nu}^{\pm}(t, \epsilon = -1) = 2e^{\pm i\frac{\pi}{2}(\frac{1}{2}-i\nu)} \frac{\Gamma(\frac{1}{2}-i\nu-iq)}{\Gamma(\frac{1}{2}-i\nu)} e^{\pi q} Q_{-\frac{1}{2}-i\nu}^{iq}(\cosh t) \quad (130)$$

Case $\epsilon = 1$. Here the integral in (129) may be split into two parts according to the sign of the expression $\cosh w \sinh \tau - \cosh \tau$: one addendum may be evaluated by means of [24] Eqs. (3.7;8) and (3.3;14) and the second by means of [24] Eqs. (3.7;5) and (3.3;13), to yield

$$\begin{aligned} f_{q,\nu}^{\pm}(\tau, \epsilon = 1) &= \frac{2ie^{\pm i\frac{\pi}{2}(\frac{1}{2}-i\nu)}}{\sinh \pi\nu} \frac{\Gamma(\frac{1}{2}-i\nu-iq)}{\Gamma(\frac{1}{2}-i\nu)} \\ &\times \left[e^{\mp \pi\nu} \cosh(\pi(q+\nu)) e^{\pi q} Q_{-\frac{1}{2}+i\nu}^{iq}(\cosh t) - \cosh \pi q e^{\pi q} Q_{-\frac{1}{2}-i\nu}^{iq}(\cosh t) \right] \end{aligned} \quad (131)$$

Expression of f in terms of Legendre functions "on the cut". There is a more elegant way and synthetic way to express the modes in terms of Legendre functions based on the use of the variable $u = Z^d$ introduced in Sect. 9.2. This alternative procedure has also the advantage to fully exhibit the underlying symmetries. The input of the relations (115) of Z into the expressions of the de Sitter waves (124) and (125) gives:

$$f_{q,\nu}^{\pm}(u, \epsilon) = \int_{-\infty}^{\infty} dv e^{-iqv} \left[(1-u^2)^{\frac{1}{2}} \cosh v \pm \epsilon i u \right]^{-\frac{1}{2}+i\nu}. \quad (132)$$

The two functions $f_{q,\nu}^{\pm}(u, \epsilon)$ are manifestly analytical in the cut plane Δ introduced in Eq. (114): the term in square brackets at the RHS vanishes for $u = \coth v$ for real v , and therefore the integral is well-defined for $u \notin (-\infty, -1) \cup (1, +\infty)$ with no additional singularity in Δ . We see once more that the

domain Δ is naturally related to the tuboids of analyticity of the de Sitter waves (120). It is therefore natural to make use of the following "Legendre function on the cut" (see App. A):

$$\mathbf{P}_{-\frac{1}{2}-i\nu}^{iq}(u) = e^{\mp \frac{1}{2}\pi q} P_{-\frac{1}{2}-i\nu}^{iq}(u), \quad (133)$$

$$\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(u) = \frac{e^{\frac{i\pi}{2}(-\frac{1}{2}+i\nu)}}{2 \sin i\pi q} \left[\frac{e^{-\frac{1}{2}\pi q} \mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)}{\Gamma(\frac{1}{2}-i\nu+iq)} - \frac{e^{\frac{1}{2}\pi q} \mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(u)}{\Gamma(\frac{1}{2}-i\nu-iq)} \right]. \quad (134)$$

The upper or lower signs of (133) refer to the imaginary part of u being positive or negative. These functions are analytic in the cut plane Δ . A brief summary of their properties and symmetries is given in Appendix (A.2). We then see, with $\mathbf{Q}_{-\frac{1}{2}+i\nu}^{*iq}(u) = \mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(-u)$, that Eqs. (130) and (131) are conveniently expressed as:

$$\begin{aligned} f_{q,\nu}^+(u, \epsilon) &= \frac{2\pi\Gamma(\frac{1}{2}-i\nu+iq)\Gamma(\frac{1}{2}-i\nu-iq)}{\Gamma(\frac{1}{2}-i\nu)} \mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(\epsilon u), \\ f_{q,\nu}^-(u, \epsilon) &= \frac{2\pi\Gamma(\frac{1}{2}-i\nu+iq)\Gamma(\frac{1}{2}-i\nu-iq)}{\Gamma(\frac{1}{2}-i\nu)} \mathbf{Q}_{-\frac{1}{2}+i\nu}^{*iq}(\epsilon u). \end{aligned} \quad (135)$$

9.6 Any dimension

Now that the two-dimensional case has been solved the general d -dimensional can be faced more easily. Let us go back to the complex time variable τ and consider say $\text{Im } \tau > 0$. By using the parametrization (4), the scalar product $X \cdot \Xi$ and the integral representation (124) may be written as follows:

$$x(r, x) \cdot a(\rho, a) \sinh \tau - \epsilon \cosh \tau = \frac{1}{\rho} \left[\frac{(x-a)^2 + r^2 + \rho^2}{2r} \sinh \tau - \epsilon \rho \cosh \tau \right], \quad (136)$$

$$\begin{aligned} (-iZ \cdot \Xi)^{-\frac{d-1}{2}+i\nu} &= (-ix \cdot a \sinh \tau + i\epsilon \cosh \tau)^{-\frac{d-1}{2}+i\nu} = \\ &= \frac{\rho^{\frac{d-1}{2}-i\nu}}{\Gamma(\frac{d-1}{2}-i\nu)} \int_0^\infty \frac{dR}{R} R^{\frac{d-1}{2}-i\nu} e^{iR \left[\frac{(x-a)^2 + r^2 + \rho^2}{2r} \sinh \tau - \epsilon \rho \cosh \tau \right]}. \end{aligned} \quad (137)$$

Given this formula, the steps to compute $F_{q,\nu}^+(\tau, \xi, a, \epsilon)$ are similar to those of the Milne case. At first the integral representations (30) and (137) are inserted into Eq. (126) and perform the Gaussian integral over x . By using the same change of variables as in (99) and (101) and identifying one factor with (the integral representation of) $(a \cdot \xi)^{-\frac{d-2}{2}-iq}$ gives

$$\begin{aligned} F_{q,\nu}^+(\tau, \xi, a, \epsilon) &= \frac{\Gamma(\frac{1}{2}-i\nu)}{\Gamma(\frac{d-1}{2}-i\nu)} \left(\frac{2\pi i}{\sinh \tau} \right)^{\frac{d-2}{2}} (a \cdot \xi)^{-\frac{d-2}{2}-iq} \times \\ &\times \int_0^\infty \frac{dR}{R} R^{-iq} \left[-i \left(\frac{R}{2} + \frac{1}{2R} \right) \sinh \tau + i\epsilon \cosh \tau \right]^{-\frac{1}{2}+i\nu}. \end{aligned} \quad (138)$$

As before, the integral at the RHS is symmetric in the exchange $q \rightarrow -q$. This result holds for $0 < \text{Im } \tau < \pi$. The other case $-\pi < \text{Im } \tau < 0$ can be obtained similarly. It follows that

$$\begin{aligned} f_{q,\nu}^\pm(\tau, \epsilon) &= \frac{\Gamma(\frac{1}{2}-i\nu)}{\Gamma(\frac{d-1}{2}-i\nu)} \left(\frac{\pm 2\pi i}{\sinh \tau} \right)^{\frac{d-2}{2}} \times \\ &\times \int_0^\infty \frac{dR}{R} R^{-iq} \left[\mp i \left(\frac{R}{2} + \frac{1}{2R} \right) \sinh \tau \pm i\epsilon \cosh \tau \right]^{-\frac{1}{2}+i\nu} \end{aligned} \quad (139)$$

where f^+ is defined in the strip $0 < \text{Im } \tau < \pi$ while f^- is defined in the strip $-\pi < \text{Im } \tau < 0$.

The inversion formula gives the expansion that we have worked so hard to obtain:

$$(\mp i Z \cdot \Xi)^{-\frac{d-1}{2}+i\nu} = \int_0^\infty n(q) dq \int d\mu(\xi) f_{q,\nu}^\pm(\tau, \epsilon) (a \cdot \xi)^{-\frac{d-2}{2}-iq} (x \cdot \xi)^{-\frac{d-2}{2}+iq} \quad (140)$$

The upper or lower sign is to be taken accordingly as $\text{Im } \tau > 0$ or $\text{Im } \tau < 0$. As in the Milne case (Sect. 8.2), it is more concise to rewrite this expansion as a one-dimensional integral by means of Eq. (68) the projector $\Pi_q(a, x)$ onto the space of open waves with eigenvalue q^2 . The latter is a particular solution of the Laplace equation, which depends on the de Sitter wave parametrisation since it is labeled by a .

$$(\mp i Z \cdot \Xi)^{-\frac{d-1}{2}+i\nu} = \int_0^\infty dq f_{q,\nu}^\pm(\tau, \epsilon) \Pi_q(a, x). \quad (141)$$

Note the analogy with the Milne case: here the role of p/m is played by the parameter a .

The whole discussion of the two-dimensional case can be repeated and we get expressions for $f_{q,\nu}^\pm(u, \epsilon)$ in terms of the Legendre functions "on the cut":

$$\begin{aligned} f_{q,\nu}^+(u, \epsilon) &= (2\pi)^{\frac{d}{2}} \frac{\Gamma(\frac{1}{2} - i\nu + iq) \Gamma(\frac{1}{2} - i\nu - iq)}{\Gamma(\frac{d-1}{2} - i\nu)} (1-u^2)^{-\frac{d-2}{4}} \mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(\epsilon u) \\ f_{q,\nu}^-(u, \epsilon) &= (2\pi)^{\frac{d}{2}} \frac{\Gamma(\frac{1}{2} - i\nu + iq) \Gamma(\frac{1}{2} - i\nu - iq)}{\Gamma(\frac{d-1}{2} - i\nu)} (1-u^2)^{-\frac{d-2}{4}} \mathbf{Q}_{-\frac{1}{2}+i\nu}^{*iq}(\epsilon u) \end{aligned} \quad (142)$$

that are analytical functions defined on the cut plane Δ .

9.7 Expansion of the de Sitter Wightman function

The more elegant way to write the de Sitter two-point Wightman function of a massive Klein-Gordon field is as a superposition of the global waves (120) [30]:

$$W(Z, Z') = \gamma_{d,\nu} \int d\mu(\Xi) (iZ \cdot \Xi)^{-\frac{d-1}{2}+i\nu} (-iZ' \cdot \Xi)^{-\frac{d-1}{2}-i\nu} \quad (143)$$

$$\gamma_{d,\nu} = \frac{\Gamma(\frac{d-1}{2} + i\nu) \Gamma(\frac{d-1}{2} - i\nu)}{2(2\pi)^d}. \quad (144)$$

where $Z \in T^-$ and $Z' \in T^+$. This writing is the one that is most similar to the Fourier plane wave expansion of the two-point function of the flat case (83).

By inserting in this representation formulae (141) one gets the spectral density ρ that provide the expansion of the Wightman function in terms of the modes of the Lobatchevskian Laplace-Beltrami operator:

$$W = \int_0^\infty dq \rho(q, \cosh \tau, \cosh \tau') \Pi_q(x, x'). \quad (145)$$

with

$$\begin{aligned} \rho(q, u, u') &= \frac{\Gamma(\frac{1}{2} - i\nu - iq) \Gamma(\frac{1}{2} + i\nu - iq) \Gamma(\frac{1}{2} - i\nu + iq) \Gamma(\frac{1}{2} + i\nu + iq)}{2 (1-u^2)^{-\frac{d-2}{4}} (1-u'^2)^{-\frac{d-2}{4}}} \\ &\times \left[\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(u) \mathbf{Q}_{-\frac{1}{2}-i\nu}^{*iq}(u') + \mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(-u) \mathbf{Q}_{-\frac{1}{2}-i\nu}^{*iq}(-u') \right]. \end{aligned} \quad (146)$$

The spectral density $\rho(q, u, u')$ may also be expressed in terms of the functions $\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)$ (see Appendix (A.2)) as follows:

$$\begin{aligned} \rho(q, u, u') = & \frac{\Gamma\left(\frac{1}{2} - i\nu - iq\right)\Gamma\left(\frac{1}{2} + i\nu - iq\right)\Gamma\left(\frac{1}{2} - i\nu + iq\right)\Gamma\left(\frac{1}{2} + i\nu + iq\right)}{4\pi (1-u^2)^{-\frac{d-2}{4}}(1-u'^2)^{-\frac{d-2}{4}}} \\ & \times \left[\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(u') + \mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(-u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(-u') \right]. \end{aligned} \quad (147)$$

These formulae provide the full solution of the difficult problem of describing the quantum Klein-Gordon field on the open de Sitter universe, provided that $|\text{Im } \nu| < \frac{1}{2}$ (see Sect. 9.4). The latter requirement can be expressed as a condition on the mass parameter according to (121):

$$m > m_{cr} = \frac{d(d-2)}{4}. \quad (148)$$

This is the condition that guarantees the convergence of the integral (126). When $m < m_{cr}$ modes that are not square-integrable are necessary [9, 13, 19] for a full and correct description.

10 Conclusions

In this paper we have settled down the basic ingredients to work out quantum theories on homogeneous and isotropic spaces with negative curvature.

While the state-of-the-art may be considered to be the expansion of the eigenmodes of the Laplacian in terms of spherical harmonics (see e.g. [5]), we give here a new set of eigenmodes, based on a different decomposition. Most important, we provide also the way to deal with these new modes, that we believe is simpler than the standard approach. Simplicity is very often the source of new progress, that however we have left for future work.

The formalism we have developed here is the closest possible to the one employed in standard (text-book) quantum mechanics on the Euclidean space \mathbb{R}^3 , and is based on the suitable Fourier-type harmonic analysis in terms of the eigenmodes of the Laplacian precisely in the same way as standard quantum mechanics is based on the Fourier transform in terms of plane waves, i.e. the momentum space representation of the wavefunctions. However, even though we have restricted our attention to square-integrable functions, the absence of translation invariance renders the task considerably more complicated than in flat space. The eigenmodes of the Laplacian are labeled in the most convenient way by using [20] vectors of the cone asymptotic to the Lobatchevski space and a real number q . An important technical point consist in finding suitable integral representations expressing such family of eigenmodes. This step is in the spirit of our earlier work [21], where general embedded manifolds (branes) were studied. These integral representations render quite simple calculations which at first sight might seem intractable.

In the second part of this paper we have applied our general construction to the study of two geometries which are quite popular nowadays: the universes of Milne and de Sitter. In both cases we have displayed explicit formulae yielding the Fourier-type expansion of the relevant spacetime plane waves in terms of the eigenmodes of the Lobatchevski Laplacian. These expansions immediately provide also harmonic expansions for the corresponding Wightman functions in terms of representations of the principal series of the $SO_0(1, d-1)$ group. One important point of our treatment of the de Sitter case is that the result is obtained by working solely in the physical region covered by the open chart, in contrast with previous approaches [9, 13] that achieved a similar goal. The apparent simplicity of our approach should not send to oblivion the fact the whole construction was thought impracticable and matter of big controversies.

The quantization of theories implying the use of non-square integrable modes remains to be attacked. A clean discussion of the square-integrable case as presented here is a necessary step forward. We expect that our methods and results will allow to proceed one step further and to tackle the analysis of a larger class of non square integrable functions.

Due to their simplicity, the methods outlined in this paper set the premises of many future developments in various other, quite different, directions.

A Legendre functions "on the cut"

A.1 Legendre functions of the first kind

Let us introduce the function

$$\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u) = e^{\mp \frac{1}{2}\pi q} P_{-\frac{1}{2}+i\nu}^{iq}(u), \quad (149)$$

where the upper or lower signs refer to the imaginary part of u being positive or negative. The function \mathbf{P} is an analytic continuation of the so-called "Legendre function on the cut", originally defined for real u such that $|u| < 1$ [24], to the whole cut-plane Δ introduced in (114). $\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)$ is an entire function of the complex parameters q and ν (the only singularities being at infinity). The functions $\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(-u)$ and $\mathbf{P}_{-\frac{1}{2}+i\nu}^{*iq}(u) = \mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(u)$ are two other solutions⁷ of the Legendre equation analytic on the same domain Δ ; one has the following relation:

$$\frac{1}{\Gamma(iq)\Gamma(1-iq)} \mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(-u) = \frac{1}{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)} \mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u) - \frac{1}{\Gamma(\frac{1}{2}-i\nu-iq)\Gamma(\frac{1}{2}+i\nu-iq)} \mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(u). \quad (150)$$

The following formula is useful to derive the spectral density of the de Sitter case and can be obtained from the relation (150):

$$\frac{\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(-u')}{\Gamma(\frac{1}{2}+iq+i\nu)\Gamma(\frac{1}{2}+iq-i\nu)} = \frac{\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(-u')}{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)} + \frac{\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(u')}{\Gamma(iq)\Gamma(1-iq)}, \quad (151)$$

or

$$\frac{\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(-u')}{\Gamma(\frac{1}{2}-iq+i\nu)\Gamma(\frac{1}{2}-iq-i\nu)} = \frac{\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(-u')}{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)} - \frac{\mathbf{P}_{-\frac{1}{2}+i\nu}^{iq}(-u)\mathbf{P}_{-\frac{1}{2}+i\nu}^{-iq}(-u')}{\Gamma(iq)\Gamma(1-iq)}. \quad (152)$$

A.2 Legendre functions of the second kind

Let us define the function $\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(u)$ by the following integral representation:

$$\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(u) = \frac{\Gamma(\frac{1}{2}-i\nu)}{2\pi\Gamma(\frac{1}{2}-i\nu+iq)\Gamma(\frac{1}{2}-i\nu-iq)} \int_{-\infty}^{\infty} dv e^{-iqv} \left[(1-u^2)^{\frac{1}{2}} \cosh v + iu \right]^{-\frac{1}{2}+i\nu}. \quad (153)$$

The function $\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(u)$ is readily seen to be a solution of the Legendre equation. It is manifestly analytic the cut-plane Δ and invariant under the change $q \rightarrow -q$. It is proportional to the standard Legendre function $Q_{-\frac{1}{2}+i\nu}^{iq}(u)$ [24] in the upper u half-plane:

$$\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(u) = \frac{e^{-i\frac{\pi}{2}(\frac{1}{2}-i\nu)}}{\pi\Gamma(\frac{1}{2}-i\nu+iq)} e^{\pi q} Q_{-\frac{1}{2}-i\nu}^{iq}(u), \quad \text{Im } u > 0; \quad (154)$$

note however that the restriction of $\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(u)$ to real u such that $|u| < 1$ does not coincide with the "Legendre function on the cut" of the second kind as defined in [24]. Obviously, \mathbf{Q} may be expressed in

⁷We define $\chi^*(u) = \overline{\chi(\bar{u})}$ where the bar denotes complex conjugation.

terms of any two independent functions of the first kind and indeed, by using [24] Eq. (3.7;24), there follows that

$$\begin{aligned}
\mathbf{Q}_{-\frac{1}{2}-i\nu}^{iq}(\pm u) &= \frac{e^{\mp i\frac{\pi}{2}(\frac{1}{2}-i\nu)}}{2\sin i\pi q} \left[\frac{e^{\mp \frac{1}{2}\pi q}}{\Gamma(\frac{1}{2}-i\nu+iq)} \mathbf{P}^{iq}_{-\frac{1}{2}+i\nu}(u) - \frac{e^{\pm \frac{1}{2}\pi q}}{\Gamma(\frac{1}{2}-i\nu-iq)} \mathbf{P}^{-iq}_{-\frac{1}{2}+i\nu}(u) \right] = \\
&= \frac{e^{\pm i\frac{\pi}{2}(\frac{1}{2}-i\nu)}}{2\sin i\pi q} \left[\frac{e^{\pm \frac{1}{2}\pi q}}{\Gamma(\frac{1}{2}-i\nu+iq)} \mathbf{P}^{iq}_{-\frac{1}{2}+i\nu}(-u) - \frac{e^{\mp \frac{1}{2}\pi q}}{\Gamma(\frac{1}{2}-i\nu-iq)} \mathbf{P}^{-iq}_{-\frac{1}{2}+i\nu}(-u) \right] \quad (155)
\end{aligned}$$

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